Exercise A.1. Suppose that $\lambda_1\lambda_2 \geq 0$ and that $U \subset X$ is a convex subset of a vector space X. Show that:

$$\lambda_1 U + \lambda_2 U = (\lambda_1 + \lambda_2)U.$$

Exercise A.2. a) Suppose that (S, τ) is a topological space, and that β is a base for τ. Show that:

i) If $x \in S$, then there exists some $B \in \beta$ with $x \in B$.

ii) If B_1, $B_2 \in \beta$, then for every $x \in B_1 \cap B_2$ there exists $B \in \beta$ with:

$$x \in B \qquad B \subset B_1 \cap B_2.$$

b) Conversely, suppose that one is given a set S and a collection β of subsets of S satisfying $i)$, $ii)$ above. Define τ by:

$$U \in \tau \iff \text{for all } x \in U, \text{ there exists } B \in \beta \text{ such that } x \in B \text{ and } B \subset U.$$

i.e. τ is the set of all unions of elements of β. Show that (S, τ) is a topological space, with base β. We say that τ is the topology generated by β

c) Suppose that β, β' both satisfy conditions $i)$, $ii)$ above and generate topologies τ, τ' respectively. Moreover, suppose that if $B \in \beta$ then for every $x \in B$ there exists $B' \in \beta'$ satisfying

$$x \in B', \qquad \text{and} \qquad B' \subset B$$

Then $\tau \subset \tau'$.

Exercise A.3. Suppose (S_1, τ_1), (S_2, τ_2) and (S_3, τ_3) are topological spaces, and that $f : S_1 \times S_2 \to S_3$ is a continuous map. Show that for each $a \in S_1$ and $b \in S_2$, the maps

$$f_a : \begin{array}{l} S_2 \to S_3, \\ y \mapsto f(a, y), \end{array} \qquad f^b : \begin{array}{l} S_1 \to S_3, \\ x \mapsto f(x, b), \end{array}$$

are continuous.

The condition that f is continuous with respect to the product topology is sometimes called *joint continuity*, while the continuity of f_a, f^b is called *separate continuity*. Thus joint continuity implies separate continuity. The converse is not true.

Exercise A.4. Show that the base

$$\beta_{\mathbb{Q}} = \{(p, q) : p, q \in \mathbb{Q},\ p < q\},$$

generates the standard topology on $\mathbb{R}$.

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Exercise A.5. Suppose that (S, d) is a metric space. Show that S is Hausdorff with respect to the metric topology.

Exercise A.6. Let us take $X = \mathbb{R}^n$, thought of as a vector space over $\mathbb{R}$ and define:

$$||(x_1, \ldots, x_n)||_p = (|x_1|^p + \ldots |x_n|^p)^{\frac{1}{p}}, \qquad p \geq 1.$$

a) Show that $(\mathbb{R}^n, ||\cdot||_p)$ is a normed vector space:

i) First check that the positivity and homogeneity property are satisfied.

ii) Establish the triangle inequality for the special case $p = 1$.

iii) Next prove Young's inequality: if $a, b \in \mathbb{R}_+$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$ then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Hint: set $t = p^{-1}$, *consider the function* $\log\left[ta^p + (1-t)b^q\right]$ *and use the concavity of the logarithm*

iv) With $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, show that if $||x||_p = 1$ and $||y||_q = 1$ then

$$\sum_{i=1}^{n} |x_i y_i| \leq 1.$$

Deduce Hölder's inequality:

$$\sum_{i=1}^{n} |x_i y_i| \leq ||x||_p \, ||y||_q, \qquad \text{for all } x, y, \in \mathbb{R}^n.$$

v) Show that

$$||x + y||_p^p \leq \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1}$$

vi) Apply Hölder's inequality to deduce:

$$||x + y||_p^p \leq \left(||x||_p + ||y||_p\right) ||x + y||_p^{p-1}$$

and conclude

$$||x + y||_p \leq ||x||_p + ||y||_p \,.$$

b) Show that the metric topology of $(\mathbb{R}^n, ||\cdot||_p)$ agrees with the standard topology.
Hint: Use part c) of Exercise A.2

Exercise A.7 $(\star)$**.** Let $X = C[0, 1]$, the set of continuous functions on the closed interval $[0, 1]$. For $f \in X$, $p \geq 0$ define:

$$||f||_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{\frac{1}{p}}$$

a) Show that X is a vector space over $\mathbb{R}$, where scalar multiplication and vector addition are defined pointwise.

b) Establish Hölder's inequality:

$$||fg||_1 \leq ||f||_p \, ||g||_q$$

for $p, q > 1$ with $p^{-1} + q^{-1} = 1$.

c) Show that $(X, ||\cdot||_p)$ is a normed space.

d) Suppose $p \leq p'$. Show that:

$$||f||_p \leq ||f||_{p'}$$

e) Let τ_p be the metric topology of $(X, ||\cdot||_p)$. Show that if $p \leq p'$:

$$\tau_p \subset \tau_{p'}.$$

f) Consider the sequence of functions:

$$f_n(x) = \begin{cases} n^{\gamma-1} & 0 \leq x < \frac{1}{n} \\ \frac{1}{n} x^{-\gamma} & \frac{1}{n} \leq x \leq 1 \end{cases}$$

where $n = 1, 2, \ldots$

i) Show that $f_n \in C[0,1]$ and

$$\lim_{n \to \infty} ||f_n||_p = \begin{cases} 0 & \gamma < \frac{p+1}{p} \\ \left(\frac{p+1}{p}\right)^{\frac{1}{p}} & \gamma = \frac{p+1}{p} \\ \infty & \gamma > \frac{p+1}{p} \end{cases}$$

ii) By choosing γ carefully, show that if $p < p'$ then

$$\tau_{p'} \not\subset \tau_p.$$

Hint: in parts b), c) follow the same steps as for the finite dimensional case in Exercise A.6.

Exercise A.8. Verify that if (D, s) is a metric space, then the metric topology defines the same notions of convergence and continuity as the standard definitions for a metric space.

Exercise A.9. Let (X, τ) be a topological vector space

a) Show that if $(x_n)_{n=1}^{\infty}$ is a τ-Cauchy sequence, then $\{x_n\}_{n=1}^{\infty}$ is bounded.

b) Fix a local base $\dot{\beta}$. Show that a sequence $(x_n)_{n=1}^{\infty}$ is τ-Cauchy if and only if for any $B \in \dot{\beta}$ we can find an integer N such that

$$x_n - x_m \in B, \qquad \text{for all } n, m \geq N.$$