Exercise 5.1. Suppose $v \in \mathscr{E}'(\mathbb{R}^n)$ and let:

$$u = \sum_{g \in \mathbb{Z}^n} \tau_g v.$$

Show that if $\phi \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp} \phi \subset K$ for some compact $K \subset \mathbb{R}^n$ then

$$u[\phi] = \sum_{g \in A} \tau_g v[\phi],$$

for some finite set $A \subset \mathbb{Z}^n$ which depends only on K. Deduce that u defines a distribution.

Exercise 5.2. Recall that for $x \in \mathbb{R}^n$:

$$||x||_1 := \sum_{i=1}^{n} |x_i|.$$

For $k \in \mathbb{N}$ set:

$$Q_k = \left\{ g \in \mathbb{Z}^n : k - \frac{1}{2} \leq ||g||_1 < k + \frac{1}{2} \right\}$$

a) Show that:

$$\#Q_k = (2k+1)^n - (2k-1)^n$$

so that $\#Q_k \leq c(1+k)^{n-1}$ for some $c > 0$.

b) By applying the Cauchy-Schwartz identity to estimate $a \cdot b$ for $a = (1, \ldots, 1)$ and $b = (|g_1|, \ldots, |g_n|)$, deduce that:

$$||g||_1 \leq \sqrt{n}\, |g|$$

c) Show that there exists a constant $C > 0$, depending only on n such that:

$$\sum_{g \in \mathbb{Z}^n; ||g||_1 \leq K} \frac{1}{(1+|g|)^{n+1}} \leq 1 + C \sum_{k=1}^{K} \frac{1}{k^2}$$

holds for all $K \in \mathbb{N}$. Deduce that:

$$\sum_{g \in \mathbb{Z}^n} \frac{1}{(1+|g|)^{n+1}} < \infty.$$

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Exercise 5.3. Show that if c_g satisfy:

$$|c_g| \leq K(1+|g|)^N$$

for some $K > 0$ and $N \in \mathbb{N}$, then:

$$\sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g}$$

converges in $\mathscr{S}'$.

Exercise 5.4. Suppose $f \in L^p_{loc.}(\mathbb{R}^n)$ is a periodic function. Fix $\epsilon > 0$, and let:

$$Q = \{x \in \mathbb{R}^n : |x_j| < 1, j = 1, \ldots, n\}, \qquad q = \left\{x \in \mathbb{R}^n : |x_j| < \frac{1}{2}, j = 1, \ldots, n\right\}$$

a) Show that there exists $h_\epsilon \in C^\infty(\mathbb{R}^n)$ with:

$$\operatorname{supp} h_\epsilon \subset Q$$

such that:

$$||f\mathbb{1}_q - h_\epsilon||_{L^p(\mathbb{R}^n)} < \epsilon.$$

Define

$$f_\epsilon = \sum_{g \in \mathbb{Z}^n} \tau_g h_\epsilon$$

b) Show that f_ϵ is smooth and periodic.

c) Show that there exists a constant c_n depending only on n such that:

$$||f - f_\epsilon||_{L^p(q)} < c_n \epsilon.$$

Exercise 5.5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = x \quad \text{for} \quad |x| < \frac{1}{2}, \qquad f(x+1) = f(x).$$

Show that:

$$f(x) = \sum_{n \in \mathbb{Z}, n \neq 0} \frac{i(-1)^n}{2\pi n} e^{2\pi i n x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(2\pi n x),$$

with convergence in $L^2_{loc.}(\mathbb{R})$.

Exercise 5.6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = \begin{cases} -1 & -\frac{1}{2} < x \leq 0 \\ 1 & 0 < x \leq \frac{1}{2} \end{cases}, \qquad f(x+1) = f(x).$$

a) Show that:

$$f(x) = \frac{1}{\pi i} \sum_{n=-\infty}^{\infty} \frac{2}{2n+1} e^{2\pi i(2n+1)x} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left[2\pi(2n+1)x\right]$$

With convergence in $L^2_{loc.}(\mathbb{R}^n)$.

Define the partial sum:

$$S_N(x) = 8 \sum_{n=0}^{N-1} \frac{1}{2\pi(2n+1)} \sin\left[2\pi(2n+1)x\right].$$

b) Show that:

$$S_N(x) = 8 \int_0^x \sum_{n=0}^{N-1} \cos\left[2\pi(2n+1)t\right] dt.$$

c) Show that:

$$\cos\left[2\pi(2n+1)t\right] \sin 2\pi t = \frac{1}{2} \left(\sin\left[2\pi(2n+2)t\right] - \sin\left[4\pi n t\right]\right)$$

And deduce:

$$S_N(x) = 8 \int_0^x \frac{\sin 4\pi N t}{2 \sin 2\pi t} dt.$$

d) Show that the first local maximum of S_N occurs at $x = \frac{1}{4N}$, and:

$$S_N\left(\frac{1}{4N}\right) \geq 8 \int_0^{\frac{1}{4N}} \frac{\sin 4\pi N t}{4\pi t} dt = \frac{2}{\pi} \int_0^{\pi} \frac{\sin s}{s} ds \simeq 1.179\ldots$$

e) Conclude that the sum in part *a*) does not converge uniformly.

This lack of uniform convergence of a Fourier series at a point of discontinuity is known as Gibbs Phenomenon.

Exercise 5.7. (*) Suppose that $\lambda = \{\lambda_1, \ldots \lambda_n\}$ is a basis for $\mathbb{R}^n$. We define the lattice generated by λ to be:

$$\Lambda = \left\{ \sum_{j=1}^{n} z_j \lambda_j : z_j \in \mathbb{Z} \right\}.$$

Define the fundamental cell:

$$q_\Lambda = \left\{ \sum_{j=1}^{n} x_j \lambda_j : |x_j| < \frac{1}{2} \right\}.$$

We say that $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic if:

$$\tau_g u = u \qquad \text{for all } g \in \Lambda.$$

a) Show that there exists $\psi \in C_0^\infty(2q_\Lambda)$ such that $\psi \geq 0$ and

$$\sum_{g \in \Lambda} \tau_g \psi = 1.$$

b) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic and ψ, ψ' are both as in part a), then

$$\frac{1}{|q_\Lambda|} u[\psi] = \frac{1}{|q_\Lambda|} u[\psi'] =: M(u)$$

c) Define the *dual lattice* by:

$$\Lambda^* := \{x \in \mathbb{R}^n : g \cdot x \in 2\pi\mathbb{Z}, \ \forall g \in \Lambda\}$$

Show that there exists a basis $\lambda^* = \{\lambda_1^*, \dots \lambda_n^*\}$ such that $\lambda_j^* \cdot \lambda_k = \delta_{jk}$, and Λ^* is the lattice induced by λ^*.

d) Show that if $g \in \Lambda^*$ then e_g is $\Lambda-$periodic.

e) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic, then:

$$\hat{u} = \sum_{g \in \Lambda^*} c_g \delta_g$$

for some $c_g \in \mathbb{C}$ satisfying $|c_g| \leq K(1 + |g|)^N$ for some $K > 0$, $N \in \mathbb{Z}$.

f) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic, then:

$$u = \sum_{g \in \Lambda^*} d_g T_{e_g}$$

where $|d_g| \leq K(1 + |g|)^N$ for some $K > 0$, $N \in \mathbb{Z}$ are given by:

$$d_g = M(e_{-g} u)$$

Exercise 5.8. Suppose $s \geq 0$.

a) Show that $\mathscr{S} \subset H^s(\mathbb{R}^n)$.

b) Suppose $f \in H^s(\mathbb{R}^n)$. Show that given $\epsilon > 0$ there exists $f_\epsilon \in \mathscr{S}$ with:

$$||f - f_\epsilon||_{H^s(\mathbb{R}^n)} < \epsilon.$$

Hint: First find $g_\epsilon \in \mathscr{S}$ such that

$$\left|\left|(\hat{f} - g_\epsilon)(1 + |\xi|)^s\right|\right|_{L^2(\mathbb{R}^n)} < \epsilon.$$

c) Show that

$$||f||_{H^s(\mathbb{R}^n)} \leq ||f||_{H^t(\mathbb{R}^n)}$$

for $t \geq s$. Deduce that:

$$||f||_{L^2(\mathbb{R}^n)} \leq \frac{1}{(2\pi)^n} ||f||_{H^s(\mathbb{R}^n)}$$

Hint: Use Parseval's formula

d) Show that the derivative D^α is a bounded linear map from $H^{s+k}(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$, where $k = |\alpha|$.

Exercise 5.9. Suppose that $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and that $u(t,x)$ is the solution of the heat equation with initial data u_0. Explicitly, u is given by:

$$u(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{-t|\xi|^2} e^{i\xi\cdot x} d\xi,$$

for $t > 0$.

a) Show that:

$$||u(t,\cdot)||_{L^2(\mathbb{R}^n)} \leq ||u_0||_{L^2(\mathbb{R}^n)},$$

b) Show that:

$$u(t,x) = u_0 \star K_t(x)$$

where the *heat kernel* is given by:

$$K_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

c) Suppose that $u_0 \geq 0$. Show that $u \geq 0$, and:

$$||u(t,\cdot)||_{L^1(\mathbb{R}^n)} = ||u_0||_{L^1(\mathbb{R}^n)}.$$

[Hint: Lemma 1.9 may be useful]

Exercise 5.10. Consider the Schrödinger equation:

$$\begin{cases} u_t = i\Delta u & \text{in } (0,T) \times \mathbb{R}^n, \\ u = u_0 & \text{on } \{0\} \times \mathbb{R}^n \end{cases} \tag{3}$$

Suppose $u_0 \in H^2(\mathbb{R}^n)$.

a) Show that (4) admits a unique solution u such that

$$\boldsymbol{u} \in C^0([0,T); H^2(\mathbb{R}^n)) \cap C^1((0,T); L^2(\mathbb{R}^n)),$$

whose spatial Fourier-Plancherel transform is given by:

$$\hat{u}(t,\xi) = \hat{u}_0(\xi) e^{-it|\xi|^2}.$$

b) Show that:

$$||u(t,\cdot)||_{H^2(\mathbb{R}^n)} = ||u_0||_{H^2(\mathbb{R}^n)}$$

*c) For $t>0$, let $K_t \in L^1_{loc.}(\mathbb{R}^n)$ be given by:

$$K_t(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{\frac{i|x|^2}{4t}},$$

where for n odd we take the usual branch cut so that $i^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$. For $\epsilon > 0$ set $K_t^\epsilon(x) = e^{-\epsilon|x|^2} K_t(x)$.

i) Show that $T_{K_t^\epsilon} \to T_{K_t}$ in $\mathscr{S}'$ as $\epsilon \to 0$.

ii) Show that if $\Re(\sigma) > 0$, then:

$$\int_{\mathbb{R}} e^{-\sigma x^2 - ix\xi} dx = \sqrt{\frac{\pi}{\sigma}} e^{-\frac{\xi^2}{4\sigma}}.$$

iii) Deduce that

$$\widehat{K_t^\epsilon}(\xi) = \left(\frac{1}{1+4it\epsilon}\right)^{\frac{n}{2}} e^{\frac{-it|\xi|^2}{1+4it\epsilon}}$$

iv) Conclude that:

$$\widehat{T_{K_t}} = T_{\tilde{K}_t},$$

where $\tilde{K}_t = e^{-it|\xi|^2}$.

*d) Suppose that $u \in \mathscr{S}(\mathbb{R}^n)$. Show that for $t>0$:

$$u(t,x) = \int_{\mathbb{R}^n} u_0(y) K_t(x-y) dy,$$

and deduce:

$$\sup_{t>0, x\in\mathbb{R}^n} |u(t,x)| \le \frac{1}{(4\pi t)^{\frac{n}{2}}} ||\hat{u}_0||_{L^1(\mathbb{R}^n)}.$$

This type of estimate which shows us that (locally) solutions to the Schrödinger equation decay in time is known as a *dispersive estimate.*

Exercise 5.11. Let $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{0\}$, $S_{*,T} := (-T,T) \times \mathbb{R}^3_*$ and $|x| = r$. You may assume the result that if $u = u(r,t)$ is radial, we have

$$\Delta u(|x|, t) = \Delta u(r,t) = \frac{\partial^2 u}{\partial r^2}(r,t) + \frac{2}{r}\frac{\partial u}{\partial r}(r,t)$$

a) Suppose $u(x,t) = \frac{1}{r} v(r,t)$ for some function v. Show that u solves the wave equation on $\mathbb{R}^3_* \times (0,T)$ if and only if v satisfies the one-dimensional wave equation

$$-\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial r^2} = 0$$

on $(0,\infty) \times (-T,T)$.

b) Suppose $f, g \in C_c^2(\mathbb{R})$. Deduce that

$$u(x,t) = \frac{f(r+t)}{r} + \frac{g(r-t)}{r}$$

is a solution of the wave equation on $S_{*,T}$ which vanishes for large $|x|$.

c) Show that if $f \in C_c^3(\mathbb{R})$ is an odd function (i.e. $f(s) = -f(-s)$ for all s) then

$$u(x,t) = \frac{f(r+t) + f(r-t)}{2r}$$

extends as a C^2 function which solves the wave equation on $S_T := (-T,T) \times \mathbb{R}^3$, with

$$u(0,t) = f'(t).$$

*d) By considering a suitable sequence of functions f, or otherwise, deduce that there exists no constant C independent of u such that the estimate

$$\sup_{S_T} (|u| + |u_t|) \le C \sup_{\Sigma_0} (|u| + |u_t|)$$

holds for all solutions $u \in C^2(S_T)$ of the wave equation which vanish for large $|x|$.