Exercise 4.1. Consider the following ODE problem. Given $f : \mathbb{R} \to \mathbb{C}$, find ϕ such that:

$$-\phi'' + \phi = f. \tag{1}$$

a) Show that if $f \in \mathscr{S}$, there is a unique $\phi \in \mathscr{S}$ solving (1), and give an expression for $\hat{\phi}$.

b) Show that

$$\phi(x) = \int_{\mathbb{R}} f(y)G(x-y)dy$$

where

$$G(x) = \begin{cases} \frac{1}{2}e^x & x < 0, \\ \frac{1}{2}e^{-x} & x \geq 0. \end{cases}$$

Exercise 4.2. Suppose $f \in L^1(\mathbb{R}^3)$ is a radial function, i.e. $f(Rx) = f(x)$, whenever $R \in SO(3)$ is a rotation.

a) Show that $\hat{f}$ is radial.

b) Suppose that $\xi = (0, 0, \zeta)$. By writing the Fourier integral in polar coordinates, show that

$$\hat{f}(\xi) = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(r)e^{-i\zeta r \cos\theta} r^2 \sin\theta d\theta dr d\phi.$$

c) Making the substitution $s = \cos\theta$, and using the fact that $\hat{f}$ is radial, deduce:

$$\hat{f}(\xi) = 4\pi \int_0^{\infty} f(r) \frac{\sin r|\xi|}{r|\xi|} r^2 dr$$

for any $\xi \in \mathbb{R}^n$.

Exercise 4.3. (*) Suppose that $f, g \in L^2(\mathbb{R}^n)$, and denote the Fourier-Plancherel transform by $\overline{\mathcal{F}}$. You may assume any results already established for the Fourier transform.

a) Show that

$$(f, g) = \frac{1}{(2\pi)^n} \left(\overline{\mathcal{F}}[g], \overline{\mathcal{F}}[g]\right).$$

b) Recall that $\check{f}(y) = f(-y)$. Show that:

$$\overline{\mathcal{F}}\left[\overline{\mathcal{F}}[f]\right] = (2\pi)^n \check{f}.$$

Hence, or otherwise, deduce that $\overline{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bijection, and that $\overline{\mathcal{F}}^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear map.

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Questions marked (*) are not assessed

c) Show that:

$$\overline{\mathcal{F}}[f](\xi) = \lim_{R\to\infty} \int_{B_R(0)} f(x)e^{-ix\cdot\xi}dx$$

with convergence in the sense of $L^2(\mathbb{R}^n)$.

d) Suppose that $f \in C^1(\mathbb{R}^n)$ and $f, D_j f \in L^2(\mathbb{R}^n)$. Show that $\xi_j \overline{\mathcal{F}}[f](\xi) \in L^2(\mathbb{R}^n)$ and:

$$\overline{\mathcal{F}}[D_j f](\xi) = i\xi_j \overline{\mathcal{F}}[f](\xi)$$

e) For $x \in \mathbb{R}$ let:

$$f(x) = \frac{\sin x}{x}$$

i) Show that $f \in L^2(\mathbb{R})$.

ii) Show that:

$$\overline{\mathcal{F}}[f](\xi) = \begin{cases} \pi & -1 < \xi < 1, \\ 0 & |\xi| \geq 1. \end{cases}$$

f) i) Show that for all $x \in \mathbb{R}^n$:

$$|f \star g(x)| \leq ||f||_{L^2(\mathbb{R}^n)} \, ||g||_{L^2(\mathbb{R}^n)} \, .$$

ii) Show that $f \star g \in C^0(\mathbb{R}^n)$ and:

$$f \star g = \mathcal{F}^{-1}\Big[\overline{\mathcal{F}}[f] \cdot \overline{\mathcal{F}}[g]\Big]$$

where:

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi\cdot x} d\xi.$$

[Hint for parts a), b), d), f): approximate by Schwartz functions]

Exercise 4.4. Work in $\mathbb{R}^3$. For $k > 0$, define the function:

$$G(x) = \frac{e^{-k|x|}}{4\pi\,|x|}$$

a) Show that $G \in L^1(\mathbb{R}^3)$.

b) Show that:

$$\hat{G}(\xi) = \frac{1}{|\xi|^2 + k^2}$$

[Hint: use Exercise 4.2, part c)]

Exercise 4.5. Consider the inhomogeneous Helmholtz equation on $\mathbb{R}^3$:

$$-\Delta\phi + k^2\phi = f \tag{2}$$

where $f \in \mathscr{S}$. Show that there exists a unique $\phi \in \mathscr{S}$ satisfying (2) given by:

$$\phi(x) = \int_{\mathbb{R}^3} f(y)G(x-y)dy,$$

where

$$G(x) = \frac{e^{-k|x|}}{4\pi |x|}.$$

[Hint: first derive an equation satisfied by $\hat{\phi}$*]*

Exercise 4.6. Verify that if $f \in L^1_{loc.}$ is such that $T_f \in \mathscr{S}'$, then:

$$\tau_x T_f = T_{\tau_x f}, \quad \text{and} \quad \check{T}_f = T_{\check{f}}$$

Exercise 4.7. Let $f : \mathbb{R} \to \mathbb{R}$ be the sign function

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and define $f_R(x) = f(x)\mathbb{1}_{[-R,R]}(x)$.

a) Sketch $f_R(x)$.

b) Show that:

$$T_{f_R} \to T_f \text{ in } \mathscr{S}' \text{ as } R \to \infty.$$

c) Show that:

$$\hat{f}_R(\xi) = 2i\frac{\cos R\xi - 1}{\xi}$$

d) For $\phi \in \mathscr{S}$, show that:

$$T_{\hat{f}_R}[\phi] = -2i\int_0^\infty \frac{\phi(x) - \phi(-x)}{x}dx + 2i\int_0^\infty \left(\frac{\phi(x) - \phi(-x)}{x}\right)\cos Rx dx$$

e) By applying the Riemann-Lebesgue Lemma, or otherwise, show that for any $\psi \in \mathscr{S}$:

$$\int_0^\infty \psi(x)\cos Rx dx \to 0$$

as $R \to \infty$.

f) Deduce that

$$\widehat{T_f} = -2iP.V.\left(\frac{1}{x}\right)$$

g) Write down $\widehat{T_H}$, where H is the Heaviside function:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$