Exercise 3.1. a) Suppose that $\{\phi_j\}_{j=1}^{\infty} \subset \mathscr{E}(\Omega)$ is a sequence such that $\phi_j \to \phi$ in $\mathscr{E}(\Omega)$, and $\chi \in \mathscr{D}(\Omega)$. Show that

$$\chi\phi_j \to \chi\phi \quad \text{in } \mathscr{D}(\Omega).$$

b) Show that if $\psi \in \mathscr{E}(\Omega)$, then there exists a sequence $\{\phi_j\}_{j=1}^{\infty} \subset \mathscr{D}(\Omega)$ such that $\phi_j \to \psi$ in $\mathscr{E}(\Omega)$.
[Hint: Take an exhaustion of Ω by compact sets and apply Lemma 1.14]

Exercise 3.2. a) Prove Lemma 2.11.
[Hint: You should argue in a similar fashion to the proof of Lemma 2.9]

b) Show that we have the inclusions:

$$\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}' \subset \mathscr{D}'(\mathbb{R}^n).$$

Exercise 3.3. Let $\{a_j\}_{j=1}^{\infty} \subset \mathbb{R}$ be a sequence of real numbers. Define for $\phi \in C^{\infty}(\mathbb{R})$:

$$u[\phi] = \sum_{j=1}^{\infty} a_j \phi(j)$$

provided that the sum converges. Give necessary and sufficient conditions on a_j such that:

a) $u \in \mathscr{E}'(\mathbb{R})$.

b) $u \in \mathscr{S}'$.

c) $u \in \mathscr{D}'(\mathbb{R})$.

Exercise 3.4. For $\xi \in \mathbb{R}^n$, define $e_\xi(x) = e^{i\xi\cdot x}$. Show that $T_{e_\xi} \in \mathscr{S}'$, and that:

$$T_{e_\xi} \to 0, \qquad \text{as } |\xi| \to \infty$$

in the topology[1] of $\mathscr{S}'$.

Exercise 3.5. Calculate the Fourier transform of the following functions $f \in L^1(\mathbb{R})$:

a) $f(x) = \dfrac{\sin x}{1+x^2}$.

b) $f(x) = \dfrac{1}{\epsilon^2 + x^2}$, for $\epsilon > 0$ a constant.

c) $f(x) = \sqrt{\dfrac{\sigma}{t}} e^{-\sigma\frac{(x-y)^2}{t}}$, where $\sigma > 0$, $t > 0$ and y are constants.

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[1] This is defined precisely as the topology of $\mathscr{D}'(\Omega)$, *mutatis mutandis*.

*d) $f(x) = \dfrac{1}{\cosh x}$.

Exercise 3.6. Suppose $f \in C^1(\mathbb{R}^n)$ and that $f, D_j f \in L^1(\mathbb{R}^n)$. Fix $\epsilon > 0$. Show that there exists $f_\epsilon \in C^1_0(\mathbb{R}^n)$ such that

$$||f - f_\epsilon||_{L^1(\mathbb{R}^n)} + ||D_j f - D_j f_\epsilon||_{L^1(\mathbb{R}^n)} < \frac{\epsilon}{2}.$$

[Hint: First construct, for large R, a smooth cut-off function $\chi_R(x)$ with $\chi_R(x) = 1$ for $|x| < R$, $\chi_R(x) = 0$ for $|x| > 2R$ and $|D\chi_R(x)| < C$, where C is independent of R.]

Exercise 3.7. Suppose $f \in L^1(\mathbb{R}^n)$, with $\operatorname{supp} f \subset B_R(0)$ for some $R > 0$.

a) Show that $\hat{f} \in C^\infty(\mathbb{R}^n)$ and for any multi-index:

$$\sup_{\xi \in \mathbb{R}^n} \left| D^\alpha \hat{f}(\xi) \right| \leq R^{|\alpha|} ||f||_{L^1(\mathbb{R}^n)}$$

b) Show that $\hat{f}$ is real analytic, with an infinite radius of convergence, i.e.:

$$\hat{f}(\xi) = \sum_\alpha D^\alpha \hat{f}(0) \frac{\xi^\alpha}{\alpha!}$$

holds for all $\xi \in \mathbb{R}^n$.

c) Show that if $\hat{f}(\xi)$ vanishes on an open set, it must vanish everywhere.
[Hint: use part i) of Lemma 3.2]

You may assume the following form of Taylor's theorem. Suppose $g \in C^{k+1}(\overline{B_r(0)})$. Then for $x \in B_r(0)$:

$$g(x) = \sum_{|\alpha| \leq k} D^\alpha \hat{f}(0) \frac{\xi^\alpha}{\alpha!} + \sum_{\beta = k+1} R_\beta(x) x^\beta$$

where the remainder $R_\beta(x)$ satisfies the following estimate in $B_r(0)$:

$$|R_\beta(x)| \leq \frac{1}{\beta!} \max_{|\alpha| = |\beta|} \max_{y \in \overline{B_r(0)}} |D^\alpha g(y)| .$$

See §A.1 of the notes for notation.