Exercise 2.1 (*). Suppose that we work over $\mathbb{R}^n$ and that $f, g, h \in \mathscr{S}$.

a) Show that for any multi-index α, we have that $D^\alpha f \in L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, i.e. that

$$||D^\alpha f||_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |D^\alpha f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

b) Define

$$\begin{aligned} F \quad &: \quad \mathbb{R}^n \times \mathbb{R}^n, \\ &\quad (x, y) \mapsto f(x)g(y - x). \end{aligned}$$

Show that $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

c) For each $x \in \mathbb{R}^n$, set

$$\begin{aligned} G_x \quad &: \quad \mathbb{R}^n \times \mathbb{R}^n, \\ &\quad (y, z) \mapsto f(y)g(z)h(x - y - z). \end{aligned}$$

Show that $G_x \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

Exercise 2.2. Show that Theorem 1.10 holds under the alternative hypotheses:

a) $f \in L^1(\mathbb{R}^n)$, $g \in C^k(\mathbb{R}^n)$ with $\sup_{\mathbb{R}^n} |D^\alpha g| < \infty$ for all $|\alpha| \leq k$.

b) $f \in L^1(\mathbb{R}^n)$ with $\operatorname{supp} f$ compact, $g \in C^k(\mathbb{R}^n)$.

Exercise 2.3. a) Prove the following identities for $r, s > 0$ and $x \in \mathbb{R}^n$:

i) $B_r(x) + B_s(0) = B_{r+s}(x)$
ii) $\overline{B_r(x)} + B_s(0) = B_{r+s}(x)$
iii) $\overline{B_r(x)} + \overline{B_s(0)} = \overline{B_{r+s}(x)}$

Suppose that $A, B \subset \mathbb{R}^n$. Show that:

b) If one of A or B is open, then so is $A + B$.

c) If A and B are both bounded, then so is $A + B$.

d) If A is closed and B is compact, then $A + B$ is closed.

e) If A and B are both compact, then so is $A + B$.

Exercise 2.4. Show that if $f \in C^k_0(\mathbb{R}^n)$ and $g \in C^l_0(\mathbb{R}^n)$ then $f \star g \in C^{k+l}_0(\mathbb{R}^n)$. Conclude that $\mathscr{D}(\mathbb{R}^n)$ is closed under convolution.

Please send any corrections to c.warnick@imperial.ac.uk
Questions marked (*) are not assessed

Exercise 2.5 (*). Suppose that $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a positive simple function,

a) Show that Minkowski's integral inequality holds for the case $p = 1$:

$$\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x,y)dx \right| dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x,y)|\, dydx$$

b) Next prove Young's inequality: if $a, b \in \mathbb{R}_+$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$ then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Hint: set $t = p^{-1}$, *consider the function* $\log\left[ta^p + (1-t)b^q\right]$ *and use the concavity of the logarithm*

c) With $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, show that if $||f||_p = 1$ and $||g||_q = 1$ then

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq 1.$$

Deduce Hölder's inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \ \leq ||f||_p\, ||g||_q\, , \qquad \text{for all } f \in L^p(\mathbb{R}^n), \ \ g, \in L^q(\mathbb{R}^n).$$

d) Set $G(y) = \left(\int_{\mathbb{R}^n} F(x,y)dx\right)^{p-1}$

i) Show that if $q = \frac{p}{p-1}$:

$$||G||_{L^q(\mathbb{R}^n)} = \left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|_{L^p(\mathbb{R}^n)}^{p-1}$$

ii) Show that:

$$\left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|_{L^p(\mathbb{R}^n)}^{p} = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} G(y)F(x,y)dy \right) dx$$

iii) Applying Hölder's inequality, deduce:

$$\left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|_{L^p(\mathbb{R}^n)}^{p} \leq ||G||_{L^p(\mathbb{R}^n)} \int_{\mathbb{R}^n} ||F(x,\cdot)||_{L^p(\mathbb{R}^n)}\, dx$$

e) Deduce that Minkowski's integral inequality

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x,y)dx \right|^p dy \right]^{\frac{1}{p}} \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |F(x,y)|^p\, dy \right]^{\frac{1}{p}} dx$$

holds for any measurable function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, where $1 \leq p < \infty$.

Exercise 2.6. a) Show that δ_x, as defined in Example 7 is continuous and linear, hence a distribution. Find the order.

b) Show that T_f, as defined in Example 7 is continuous and linear, hence a distribution. Find the order.

c) By constructing a suitable sequence of smooth functions show that the order of $P.V.\left(\frac{1}{x}\right)$ is one.

Exercise 2.7. Suppose $f \in L^1_{loc.}(\Omega)$. Take ϕ_ϵ as in Theorem 1.13, and define for x with $d(x, \partial\Omega) \geq \epsilon$:

$$f_\epsilon(x) := T_f\left[\tau_x \check{\phi}_\epsilon\right].$$

Show that for any compact $K \subset \Omega$:

$$||f_\epsilon - f||_{L^1(K)} \to 0$$

as $\epsilon \to 0$.
[Hint: follow the proof of Theorem 2.2, but use part b) of Theorem 1.13]

Exercise 2.8. a) Show that if $f_1, f_2 \in C^0(\Omega)$ and $a \in C^\infty(\Omega)$, then

$$aT_{f_1} + T_{f_2} = T_{af_1+f_2}$$

b) Show that if $f \in C^k(\Omega)$ then

$$D^\alpha T_f = T_{D^\alpha f}$$

for $|\alpha| \leq k$. Deduce that $\iota \circ D^\alpha = D^\alpha \circ \iota$.

c) Deduce that if $f \in C^k(\Omega)$ then

$$\sum_{|\alpha| \leq k} a_\alpha D^\alpha T_f = T_{Lf}.$$

where

$$Lf = \sum_{|\alpha| \leq k} a_\alpha D^\alpha f$$

Exercise 2.9. a) Show that for $f, g \in C^0_0(\mathbb{R}^n)$:

$$T_{f\star g} = T_f \star T_g.$$

b) Show that convolution is linear in both of its arguments, i.e. if $u_i \in \mathscr{D}'(\mathbb{R}^n)$ and u_3, u_4 have compact support then

$$(u_1 + au_2) \star u_3 = u_1 \star u_3 + au_2 \star u_3$$

and

$$u_1 \star (u_3 + au_4) = u_1 \star u_3 + au_1 \star u_4$$

where $a \in \mathbb{C}$ is a constant.

Exercise 2.10. a) Show that if $\phi \in \mathscr{D}(\mathbb{R}^n)$ then

$$\delta_0 \star \phi = \phi$$

b) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ has compact support, then

$$\delta_0 \star u = u$$