Example Sheet 8 M2PM1: Real Analysis
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Exercise 8.1. Show that $f : \mathbb{R}^2 \to \mathbb{R}$ is everywhere differentiable, and find the differential when:

a) $f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x^2 + y^2 - x - xy$

b) $f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \dfrac{1}{\sqrt{1 + x^2 + y^2}}$

c) $f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x^5 y^2$

Exercise 8.2. Find the minimum of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by:

$$f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto x^4(y^2 + x^2) + z^2 - 4z$$

Exercise 8.3. Suppose A is a symmetric $(n \times n)$ matrix. Consider the function:

$$\begin{array}{rccl} f & : & \mathbb{R}^n & \to & \mathbb{R} \\ & & x & \mapsto & x^t A x. \end{array}$$

a) Show that f is differentiable at all points $p \in \mathbb{R}^n$, with:

$$Df(p) = 2p^t A$$

b) Find:

$$\mathrm{Hess}\, f(p).$$

Exercise 8.4 (*)**.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{xy^3 - x^3 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

a) Show that:

$$D_1 f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{y^3 - 3x^2 y}{x^2 + y^2} - \dfrac{2x\left(xy^3 - x^3 y\right)}{\left(x^2 + y^2\right)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

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Questions marked $(*)$ are optional

and

$$D_2 f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{3y^2x - x^3}{x^2+y^2} - \dfrac{2y\left(xy^3 - x^3y\right)}{\left(x^2+y^2\right)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0), \end{cases}$$

and show that these functions are both continuous at $(0,0)$.

b) Show that:

$$\lim_{t\to 0} \frac{1}{t}\left(D_1 f(te_2) - D_1 f(0)\right) = 1$$

and

$$\lim_{t\to 0} \frac{1}{t}\left(D_2 f(te_1) - D_2 f(0)\right) = -1$$

c) Conclude that both $D_2D_1f(0)$ and $D_1D_2f(0)$ exist, but that:

$$D_2D_1f(0) \neq D_1D_2f(0)$$

Exercise 8.5 (*)**.** a) By induction on $m \in \mathbb{N}$, show that if $x_1, \ldots, x_m \in \mathbb{R}$ and $j \in \mathbb{N}$ then:

$$(x_1 + \ldots + x_m)^j = \sum_{\substack{n_1,\ldots,n_m \in \mathbb{N} \\ n_1+n_2+\ldots+n_m = j}} \begin{pmatrix} j \\ n_1, n_2, \ldots, n_m \end{pmatrix} x_1^{n_1} \cdots x_m^{n_m},$$

where the multinomial coefficient is defined by:

$$\begin{pmatrix} j \\ n_1, n_2, \ldots, n_m \end{pmatrix} = \frac{j!}{n_1! n_2! \cdots n_m!}.$$

[Hint: Note that the case $m = 1$ is trivial and $m = 2$ is the binomial theorem. Work by induction using $(x_1 + \ldots + x_{m-1} + x_m)^j = (x_1 + \ldots + (x_{m-1} + x_m))^j$]

b) By writing $(x_1 + \ldots + x_m)^j = (x_1 + \ldots + x_m)(x_1 + \ldots + x_m)^{j-1}$ and expanding, show that for $j \geq 1$:

$$\begin{pmatrix} j \\ n_1, n_2, \ldots, n_m \end{pmatrix} = \frac{(j-1)!}{(n_1-1)! n_2! \cdots n_m!} + \frac{(j-1)!}{n_1!(n_2-1)! \cdots n_m!} + \ldots$$
$$\ldots + \frac{(j-1)!}{n_1! n_2! \cdots (n_m-1)!}$$

holds for any $n_1, \ldots, n_m \in \mathbb{N}$, with $n_1 + n_2 + \ldots + n_m = j$

Exercise 8.6. Let $\Omega = \{(x,y)^t \in \mathbb{R}^2 : x > 0\}$. Consider the function $f : \Omega \to \mathbb{R}^2$ given by:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \sin y \\ x \cos y \end{pmatrix}.$$

a) Show that f is differentiable at all $p = (\xi, \eta)^t \in \Omega$, with:

$$Df(p) = \begin{pmatrix} \sin\eta & \xi\cos\eta \\ \cos\eta & -\xi\sin\eta \end{pmatrix}.$$

b) Show that $Df(p)$ is invertible for all $p \in \Omega$.

c) Show that $f : \Omega \to \mathbb{R}^2$ is not injective. Deduce that the restriction to open sets U, V in the inverse function theorem is necessary.

Exercise 8.7. a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable in a neighbourhood of the origin, and $f'(0) = 0$. Give an example to show that f may nevertheless be bijective.

[Hint: Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto x^3$.]

b) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective, differentiable at the origin, and $\det Df(0) = 0$. Show that f^{-1} is not differentiable at $f(0)$.

[Hint: Assume that f^{-1} is differentiable at $f(0)$ and apply the chain rule to $\iota = f^{-1} \circ f = f \circ f^{-1}$ to derive a contradiction.]