Exercise 7.1. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by:

$$f(x) = x.$$

Show that f is differentiable at each $p \in \mathbb{R}^n$ and:

$$Df(p) = \iota,$$

where $\iota : \mathbb{R}^n \to \mathbb{R}^n$ is the identity.

Exercise 7.2. Show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x^2 + y^2,$$

is differentiable at all points $p = (\xi, \eta)^t \in \mathbb{R}^2$ with Jacobian:

$$Df(p) = (2\xi \quad 2\eta)$$

Exercise 7.3. One might hope that the differential can be calculated by finding:

$$\lim_{x \to p} \frac{f(x) - f(p)}{||x - p||}.$$

By considering the example of Exercise 7.1 or otherwise, show that this limit may not always exist, even if f is differentiable at p.

Exercise 7.4. Suppose that $\Omega \subset \mathbb{R}^n$ is open, and $f, g : \Omega \to \mathbb{R}^m$ are differentiable at $p \in \Omega$. Show that $h = f + g$ is differentiable at p and

$$Dh(p) = Df(p) + Dg(p)$$

Exercise 7.5. Suppose $\Omega, \Omega' \subset \mathbb{R}^n$ are open, $g : \Omega \to \Omega'$ and $f : \Omega' \to \Omega$ are functions such that g is differentiable at $p \in \Omega$ and f is differentiable at $g(p) \in \Omega'$ and moreover:

$$f \circ g(x) = x, \qquad \forall\ x \in \Omega.$$
$$g \circ f(x) = x, \qquad \forall\ x \in \Omega'.$$

Show that:

$$Df(g(p)) = (Dg(p))^{-1}.$$

Please send any corrections to c.warnick@imperial.ac.uk
Questions marked $(*)$ are optional

Exercise 7.6 (*). a) Show that the map $P : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$P : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto xy$$

is differentiable at each point $p = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{R}^2$, with Jacobian:

$$DP(p) = \begin{pmatrix} \eta & \xi \end{pmatrix}.$$

b) Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $q \in \mathbb{R}^n$. Show that the map $Q : \mathbb{R}^n \to \mathbb{R}^2$ given by:

$$Q : z \mapsto \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}$$

is differentiable at q and:

$$DQ(q) = \begin{pmatrix} Df(q) \\ Dg(q) \end{pmatrix}$$

c) Show that $F : \mathbb{R}^n \to \mathbb{R}$ given by $F(z) = f(z)g(z)$ for all $z \in \mathbb{R}^n$ is differentiable at q, and:

$$DF(q) = g(q)Df(q) + f(q)Dg(q)$$

[Hint: Note that $F = P \circ Q$.]