Example Sheet 6 M2PM1: Real Analysis
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Exercise 6.1. Suppose that $f : [0, \infty) \to [0, \infty)$ is a non-negative, continuous, monotone decreasing function, and let $N, M \in \mathbb{Z}$.

a) Show that:

$$\sum_{n=N}^{M} f(n) \geq \int_N^{M+1} f(x)dx.$$

[Hint: Take a uniform partition of unit length intervals, and use the monotonicity to estimate the upper sum]

b) Show that:

$$\sum_{n=N+1}^{M} f(n) \leq \int_N^{M} f(x)dx.$$

[Hint: Take a uniform partition of unit length intervals, and use the monotonicity to estimate the lower sum]

c) Establish the inequality:

$$\int_N^{M+1} f(x)dx \leq \sum_{n=N}^{M} f(n) \leq f(N) + \int_N^{M} f(x)dx$$

d) Show that

$$\sum_{n=N}^{\infty} f(n)$$

converges if and only if the improper integral:

$$\int_N^{\infty} f(x)dx$$

exists.

e) Show that the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

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Questions marked $(*)$ are optional

Exercise 6.2. Let $n \in \mathbb{N}$ with $n \geq 2$. By choosing a suitable partition to approximate the integral:

$$\int_1^n \log x \, dx$$

by upper and lower sums, show that:

$$\sum_{k=1}^{n-1} \log k \leq n \log n - n + 1 \leq \sum_{k=2}^{n} \log k.$$

Deduce that:

$$n^n e^{-n+1} \leq n! \leq (n+1)^{n+1} e^{-n}.$$

Exercise 6.3. Suppose $f : [a, b] \to \mathbb{R}^m$ is integrable.

a) Show that if $x, y \in [a, b]$:

$$\left| ||f(x)|| - ||f(y)|| \right| \leq \sum_{l=1}^{m} \left| f^l(x) - f^l(y) \right|,$$

where $f(t) = e_1 f^1(t) + e_2 f^2(t) + \cdots + e_m f^m(t)$.

[Hint: Exercise 1.1]

b) Deduce that if I is any subinterval of $[a, b]$:

$$\sup_{x \in I} ||f(x)|| - \inf_{y \in I} ||f(y)|| \leq \sum_{l=1}^{m} \left(\sup_{x \in I} f^l(x) - \inf_{y \in I} f^l(y) \right).$$

[Hint: Take the supremum over x and infimum over y in the previous inequality]

c) Prove that $||f|| : [a, b] \to \mathbb{R}$ is integrable.

[Hint: Look back at the proof of part i) of Theorem 2.8]

Exercise 6.4. Let $f : [a, b] \to \mathbb{R}^m$ be an integrable function, and write

$$f(t) = e_1 f^1(t) + e_2 f^2(t) + \cdots + e_m f^m(t)$$

for $f^i : [a, b] \to \mathbb{R}$.

a) For a tagged partition $(\mathcal{P}, \tau)$, show that:

$$R(f, \mathcal{P}, \tau) = e_1 R(f^1, \mathcal{P}, \tau) + e_2 R(f^2, \mathcal{P}, \tau) + \ldots + e_m R(f^m, \mathcal{P}, \tau)$$

b) Show that if $((\mathcal{P}_l, \tau_l))_{l=0}^{\infty}$ is a sequence of tagged partitions of $[a, b]$ with mesh $\mathcal{P}_l \to 0$, then

$$R(f, \mathcal{P}_l, \tau_l) \to \int_a^b f(x) dx,$$

as $l \to \infty$.

c) Show that for a tagged partition $(\mathcal{P}, \tau)$:

$$||R(f, \mathcal{P}, \tau)|| \leq R(||f||, \mathcal{P}, \tau).$$

d) Deduce that:

$$\left|\left|\int_a^b f(t)dt\right|\right| \leq \int_a^b ||f(t)||\, dt.$$

Note that an identical argument shows that if $f : [a, b] \to \mathbb{C}$ then:

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|\, dt.$$

Exercise 6.5 (*).

Suppose $f, g, h : [a, b] \to \mathbb{R}^m$ are integrable. We define:

$$(f, g) := \int_a^b \langle f(t), g(t) \rangle\, dt, \qquad ||f||_{L^2} = \sqrt{(f, f)}.$$

a) Show that:

$$(f, g) = (g, f), \qquad (f + h, g) = (f, g) + (h, g), \qquad (\lambda f, g) = \lambda(f, g),$$

where $\lambda \in \mathbb{R}$.

b) For $t \in \mathbb{R}$, show that:

$$||f + tg||_{L^2}^2 = ||f||_{L^2}^2 + 2t(f, g) + t^2 ||g||_{L^2}^2 \geq 0. \tag{5}$$

c) By thinking of (5) as a quadratic in t and considering its possible roots, deduce the *Cauchy-Schwartz* inequality:

$$|(f, g)| \leq ||f||_{L^2} ||g||_{L^2}.$$

When does equality hold?

d) Deduce that:

$$||f + g||_{L^2} \leq ||f||_{L^2} + ||g||_{L^2}.$$