Example Sheet 4 M2PM1: Real Analysis
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Exercise 4.1. Let $A \subset \mathbb{R}^n$, suppose $f, g : A \to \mathbb{R}$ are bounded, and let $\lambda \geq 0$. Show that:

a) $\sup_{x\in A]} -f(x) = -\inf_{x\in A} f(x)$,

b) $\inf_{x\in A} -f(x) = -\sup_{x\in A} f(x)$,

c) $\sup_{x\in A} \lambda f(x) = \lambda \sup_{x\in A} f(x)$,

d) $\sup_{x\in A} (f(x) + g(x)) \leq \sup_{x\in A} f(x) + \sup_{x\in A} g(x)$,

e) $\inf_{x\in A} (f(x) + g(x)) \geq \inf_{x\in A} f(x) + \inf_{x\in A} g(x)$,

f) $\inf_{x\in A} \lambda f(x) = \lambda \inf_{x\in A} f(x)$,

g) $\left|\sup_{x\in A} f(x)\right| \leq \sup_{x\in A} |f(x)|$,

h) $\left|\inf_{x\in A} f(x)\right| \leq \sup_{x\in A} |f(x)|$.

Exercise 4.2. Let $[a, b] \subset \mathbb{R}$ be any finite interval and $\mathcal{P}$ be any partition of $[a, b]$. Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded functions, and that $\lambda \geq 0$. Show that:

a) $U(-f, \mathcal{P}) = -L(f, \mathcal{P})$,

b) $L(-f, \mathcal{P}) = -U(f, \mathcal{P})$,

c) $U(\lambda f, \mathcal{P}) = \lambda U(f, \mathcal{P})$,

d) $L(\lambda f, \mathcal{P}) = \lambda L(f, \mathcal{P})$,

e) $U(f + g, \mathcal{P}) \leq U(f, \mathcal{P}) + U(g, \mathcal{P})$,

f) $L(f + g, \mathcal{P}) \geq L(f, \mathcal{P}) + L(g, \mathcal{P})$.

Exercise 4.3. Suppose that $\mathcal{P} \preceq \mathcal{Q}$. Show that:

$$0 \leq U(f, \mathcal{Q}) - L(f, \mathcal{Q}) \leq U(f, \mathcal{P}) - L(f, \mathcal{P})$$

Exercise 4.4. Let $A \subset [-1, 1]$ be a *finite* set, and let $\chi_A : [-1, 1] \to \{0, 1\}$ be the *characteristic function* of A. That is:

$$\chi_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

Show that χ_A is integrable, and:

$$\int_{-1}^{1} \chi_A(x)dx = 0.$$

Exercise 4.5. Suppose that $a < c < b$ and suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Let $\mathcal{P}$ be a partition of $[a, b]$ of the form:

$$\mathcal{P} = (a, x_1, \ldots, x_{l-1}, x_l = c, x_{l+1}, \ldots, x_{k-1}, b).$$

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Questions marked $(*)$ are optional

and define $\mathcal{P}_L$ and $\mathcal{P}_R$ to be partitions of $[a,c]$ and $[c,b]$ respectively, given by:

$$\mathcal{P}_L = (a, x_1, \ldots, x_{l-1}, c), \qquad \mathcal{P}_R = (c, x_{l+1}, \ldots, x_{k-1}, b).$$

Show that:

$$L(f, \mathcal{P}) = L(f|_{[a,c]}, \mathcal{P}_L) + L(f|_{[c,b]}, \mathcal{P}_R),$$
$$U(f, \mathcal{P}) = U(f|_{[a,c]}, \mathcal{P}_L) + U(f|_{[c,b]}, \mathcal{P}_R).$$

Exercise 4.6. Suppose $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ are integrable.

a) Show that:

$$(b-a) \inf_{x\in[a,b]} f(x) \leq \int_a^b f(x)dx \leq (b-a) \sup_{x\in[a,b]} f(x).$$

[Hint: Consider the trivial partition $\mathcal{P} = (a,b)$, and use Theorem 2.2]

b) Establish the estimate:

$$\left| \int_a^b f(x)dx \right| \leq (b-a) \sup_{x\in[a,b]} |f(x)|$$

[Hint: Use part a) applied to both f and $-f$]

c) Show that if $0 \leq f(x)$ for all $x \in [a,b]$ then:

$$0 \leq \int_a^b f(x)dx$$

[Hint: Use part a)]

d) Show that if $f(x) \leq g(x)$ for all $x \in [a,b]$ then:

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

[Hint: Use part c) applied to $(g-f)$.]

e) Prove that:

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|\, dx$$

[Hint: Note that $f(x) \leq |f(x)|$ and $-f(x) \leq |f(x)|$ and apply part d)]

Exercise 4.7 (*)**.** Consider Thomae's function $f : [0,1] \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & x = 0, \\ \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}, \text{ where hcf}(p,q) = 1, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Show that f is integrable.