Exercise 3.1. For $i \in \mathbb{N}$, let $f_i : [0,1] \to \mathbb{R}$ be given by:

$$f_i(x) = \begin{cases} 2^i x & 0 \le x < 2^{-i}, \\ 2 - 2^i x & 2^{-i} \le x < 2^{-(i-1)} \\ 0 & x \ge 2^{-(i-1)}, \end{cases}$$

a) Sketch f_i.

b) Show that $f_i \to 0$ pointwise.

c) Is it true that:

$$\lim_{i\to\infty} \sup_{x\in[0,1]} f_i(x) = \sup_{x\in[0,1]} \lim_{i\to\infty} f_i(x)?$$

Exercise 3.2. State whether the sequence $(f_i)_{i=0}^{\infty}$ of functions converges pointwise or uniformly, when:

a) $$\begin{array}{rcccl} f_i & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & e^x + \frac{1}{i+1} \end{array}$$

b) $$\begin{array}{rcccl} f_i & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & x + 2^{-i}x^2 \end{array}$$

c) $f_i : [0,1] \to \mathbb{R}$, where:

$$f_i(x) = \begin{cases} 2^i x & 0 \le x < 2^{-i}, \\ 2 - 2^i x & 2^{-i} \le x < 2^{-(i-1)} \\ 0 & x \ge 2^{-(i-1)}, \end{cases}$$

Exercise 3.3. Suppose $A \subset \mathbb{R}^n$ and let $(f_i)_{i=0}^{\infty}$, where $f_i : A \to \mathbb{R}^m$ be a sequence of functions. Classify whether the following statements are equivalent to

i) $f_i \to f$ pointwise,

ii) $f_i \to f$ uniformly,

iii) neither.

a) Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i \ge N$:

$$\sup_{x\in A} ||f_i(x) - f(x)|| < \epsilon.$$

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Questions marked $(*)$ are optional

b) For all $x \in A$, for all $i \in \mathbb{N}$ there exists $\epsilon > 0$ such that:

$$||f_i(x) - f(x)|| < \epsilon.$$

c) There exists $N \in \mathbb{N}$ such that for all $x \in A$, $i \geq N$, $\epsilon > 0$:

$$||f_i(x) - f(x)|| < \epsilon.$$

*d) $\forall \epsilon > 0 \, \forall x \in A \, \exists N \in \mathbb{N} \, (i \geq N \implies ||f_i(x) - f(x)|| < \epsilon)$

*e) $\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall x \in A \, (i \geq N \implies ||f_i(x) - f(x)|| < \epsilon)$

*f) $\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall x \in A \, (i \geq N \iff ||f_i(x) - f(x)|| < \epsilon)$

Exercise 3.4. Suppose $A \subset \mathbb{R}^n$ and let $(f_i)_{i=0}^{\infty}$, where $f_i : A \to \mathbb{R}^m$ be a sequence of functions. Suppose that $f_i \to f$ uniformly on A. Show that if $B \subset A$, then $f_i \to f$ uniformly on B.

Exercise 3.5 (*)**.** Let $E \subset \mathbb{R}^n$ be compact. We define $C^0(E, \mathbb{R}^m)$ to be the set of all continuous functions $f : E \to \mathbb{R}^m$, and equip $C^0(E, \mathbb{R}^m)$ with the norm:

$$\begin{array}{rccl} ||\cdot||_{C^0} & : & C^0 & \to & [0, \infty) \\ & & f & \mapsto & ||f||_{C^0} := \sup_{x \in E} ||f(x)|| \, . \end{array}$$

Show that $C^0(E, \mathbb{R}^m)$ is complete.

Exercise 3.6 (*)**.** Suppose that $f : (a, b) \to \mathbb{R}$ is uniformly continuous, and suppose $(x_j)_{j=0}^{\infty}$ and $(y_j)_{j=0}^{\infty}$ are two sequences of points $x_j, y_j \in (a, b)$ with $x_j \to a$ and $y_j \to b$.

a) Show that the sequences $(f(x_j))_{j=0}^{\infty}$ and $(f(y_j))_{j=0}^{\infty}$ are Cauchy.

b) Setting $f_L := \lim_{j\to\infty} f(x_j)$, $f_R := \lim_{j\to\infty} f(y_j)$, show that the function $\tilde{f} : [a, b] \to \mathbb{R}$ given by:

$$\tilde{f}(x) = \begin{cases} f_L & x = a, \\ f(x) & a < x < b \\ f_R & x = b, \end{cases}$$

is continuous on $[a, b]$.

c) Deduce that $f : (a, b) \to \mathbb{R}$ is uniformly continuous if and only if there exists a continuous function $\tilde{f} : [a, b] \to \mathbb{R}$ such that $f(x) = \tilde{f}(x)$ for all $x \in (a, b)$. We say that $\tilde{f}$ is an extension of f.

Exercise 3.7 (*)**.** A Fourier series is a series of the form:

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \, ,$$

where $a_k, b_k \in \mathbb{R}$. Show that if $\sum_{k=1}^{\infty} (|a_k| + |b_k|)$ converges, then the Fourier series converges uniformly on $\mathbb{R}$ to a uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying:

$$f(x + 2\pi) = f(x), \qquad \text{for all } x.$$

Exercise 3.8 (*)**.** Let $h : \mathbb{R} \to \mathbb{R}$ be the periodic function satisfying:

$$h(x) = |x| \qquad x \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$

and $h(x+1) = h(x)$.

a) Sketch $h(x)$.

b) Show that $h(x)$ is uniformly continuous on $\mathbb{R}$ and differentiable at each point $x \in \mathbb{R}$ such that $2x \notin \mathbb{Z}$.

Define the Takagi function[1] $T : \mathbb{R} \to \mathbb{R}$ by:

$$T(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} h\left(2^i x\right) \tag{4}$$

c) Show that the sum in (4) converges uniformly on $\mathbb{R}$ to a uniformly continuous function $T : \mathbb{R} \to \mathbb{R}$.

   [Hint: Apply the Weierstrass $M-$test]

d) Let $x \in \mathbb{R}$. Show that there exist unique u_n, v_n such that:

   i) $2^n v_n, 2^n u_n \in \mathbb{Z}$,

   ii) $v_n - u_n = 2^{-n}$ and,

   iii) $u_n \leq x < v_n$.

   [Hint: First show that there exist consecutive integers $\tilde{u}_n, \tilde{v}_n$ satisfying $\tilde{u}_n \leq 2^n x < \tilde{v}_n$.]

e) Let $h_i(x) = \frac{1}{2^i} h\left(2^i x\right)$. Show that

   i) $h_i(u_n) = h_i(v_n) = 0$ for $i \geq n$.
   [Hint: Note that $h_i(u_n) = 2^{-i} h(2^i u_n)$, and that $h(x)$ vanishes for $x \in \mathbb{Z}$.]

   ii) $\dfrac{h_i(v_n) - h_i(u_n)}{v_n - u_n} = \pm 1$ for $i < n$.
   [Hint: First try the case $i = n-1$, and note that $2^i u_n, 2^i v_n$ are consecutive half-integers, then try to generalise]

f) Deduce that:

$$\frac{T(v_n) - T(u_n)}{v_n - u_n} = \sum_{i=0}^{n-1} \frac{h_i(v_n) - h_i(u_n)}{v_n - u_n},$$

   and conclude that the sum on the right does *not* converge as $n \to \infty$.

   [Hint: Use the previous part, and note that an infinite sum, each term of which is ± 1 does not converge.]

[1] Sometimes called the blancmange function.

g) Show that T is not differentiable at any x.

[Hint: Suppose that T is differentiable at x, apply Taylor's Theorem about x to estimate $T(v_n) - T(u_n)$ and derive a contradiction.]

Exercise 3.9 (*). Suppose $F = (f_i)_{i=0}^{\infty}$ is a sequence of continuous functions $f_i : [a, b] \to \mathbb{R}$, which are differentiable on (a, b) and satisfy:

$$\sup_{x \in (a,b)} \left| f_i'(x) \right| \leq M$$

for some M independent of i. Show that F is uniformly equicontinuous.

[Hint: Use the mean value theorem on the interval $[x, y]$]