Exercise 2.1. We say that a sequence $(x_i)_{i=0}^\infty$ with $x_i \in \mathbb{R}^n$ is a *Cauchy sequence* if the following holds: given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i, j \geq N$ we have:

$$||x_i - x_j|| < \epsilon.$$

a) Suppose the sequence $(x_i)_{i=0}^\infty$ with $x_i \in \mathbb{R}^n$ converges to x. Show that $(x_i)_{i=0}^\infty$ is a Cauchy sequence.

[Hint: write $x_i - x_j = (x_i - x) - (x_j - x)$ and use the triangle inequality]

b) Suppose $(x_i)_{i=0}^\infty$ with $x_i \in \mathbb{R}^n$ is a Cauchy sequence. Show that $(x_i)_{i=0}^\infty$ is bounded.

[Hint: Apply the definition of a Cauchy sequence with $\epsilon = 1$ and use the triangle inequality]

c) Show that every Cauchy sequence in $\mathbb{R}^n$ converges.

[Hint: extract a convergent subsequence using Bolzano–Weierstrass, and use the Cauchy property to show convergence]

Exercise 2.2 (*). Suppose $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Show that $\lim_{x\to p} f(x) = F$ if and only if for any sequence $(x_i)_{i=0}^\infty$ with $x_i \in A, x_i \neq p$ and $x_i \to p$ we have:

$$f(x_i) \to F, \qquad \text{as } i \to \infty.$$

Exercise 2.3. a) Show that the map

$$\begin{array}{cccc} f & : & \mathbb{R} & \to & \mathbb{R}^n \\ & & x & \mapsto & (x, 0, \ldots, 0)^t \end{array}$$

is continuous on $\mathbb{R}$.

b) Let $A \subset \mathbb{R}^n$ and suppose we are given a map:

$$\begin{array}{cccc} f & : & A & \to & \mathbb{R}^m \\ & & (x^1, \ldots, x^n)^t & \mapsto & \left(f^1(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n)\right)^t. \end{array}$$

Show that f is continuous at $p \in A$ if and only if each map $f^k : A \to \mathbb{R}$ is continuous at p, for $k = 1, \ldots, m$.

c) Show that a map $f : \mathbb{R}^n \to \mathbb{R}$ which is polynomial in the coordinates is continuous on $\mathbb{R}^n$.

Exercise 2.4. Suppose that $E \subset \mathbb{R}^n$ is compact and $f : E \to \mathbb{R}^m$ is continuous on E.

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Questions marked $(*)$ are optional

a) Show that $f(E) := \{f(x) : x \in E\}$ is bounded.

[Hint: Write $f(x) = (f^1(x), \ldots, f^m(x))$ and apply Theorem I.9 to each $f^j(x)$.]

b) Show that $f(E)$ is closed, hence compact.

[Hint: Consider a sequence $(x_i)_{i=0}^{\infty}$ with $x_i \in E$ such that $f(x_i) \to F$. Extract a convergent subsequence to justify that $F = f(x)$ for some $x \in E$.]

c) Is it true that $f(E)$ is closed if we just assume E is closed (not necessarily compact)?

[Hint: Consider the function $x \mapsto \tanh x$ for $x \in \mathbb{R}$.]

Exercise 2.5 (*)**.** a) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$, and suppose $U \subset \mathbb{R}^m$ is open. Show that:

$$f^{-1}(U) := \{x \in \mathbb{R}^n : f(x) \in U\}$$

is open.

b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has the property that $f^{-1}(U) \subset \mathbb{R}^n$ is open for every open $U \subset \mathbb{R}^m$. Show that f is continuous on $\mathbb{R}^n$.

Exercise 2.6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable (hence continuous) everywhere on $\mathbb{R}$ and the derivative satisfies:

$$\left|f'(x)\right| \leq M, \qquad \text{for all } x \in \mathbb{R}.$$

a) Show that for $x, y \in \mathbb{R}$ we have the estimate:

$$|f(x) - f(y)| \leq M\,|x - y|\,. \tag{3}$$

[Hint: Use the mean value theorem]

b) Deduce that f is uniformly continuous on $\mathbb{R}$.

The condition (3), which can be imposed whether or not f is differentiable, is known as the Lipschitz condition.

Exercise 2.7. Which of the following functions is i) continuous, ii) uniformly continuous on its domain?

a) $$\begin{array}{rcccc} f & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & e^x \end{array}$$

b) $$\begin{array}{rcccc} f & : & (0,1) & \to & \mathbb{R} \\ & & x & \mapsto & e^x \end{array}$$

c) $$\begin{array}{rcccc} f & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & \sin x \end{array}$$

d) $$\begin{array}{rcccc} f & : & [0,\infty) & \to & \mathbb{R} \\ & & x & \mapsto & \sqrt{x} \end{array}$$

*f)
$$\begin{array}{rcccl} f & : & \mathbb{R}^n \setminus \{0\} & \to & \mathbb{R}^n \\ & & x & \mapsto & \frac{x}{||x||} \end{array}$$

Exercise 2.8. Let $A \subset \mathbb{R}^n$ and suppose that $f, g : A \to \mathbb{R}$.

a) Show that if f, g are uniformly continuous and bounded, then so is fg.

[Hint: Write

$$f(x)g(x)-f(y)g(y) = [f(x) - f(y)]\,[g(x) - g(y)]+[f(x) - f(y)]\,g(y)+f(y)\,[g(x) - g(y)]$$

and use the triangle inequality]

b) Show that if f, g are uniformly continuous but not necessarily bounded, then fg need not be uniformly continuous.

[Hint: Consider $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ and $g(x) = \sin x$]