Exercise 1.1. a) Show that the inner product has the following properties:

$$\langle x, y \rangle = \langle y, x \rangle, \qquad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \qquad \langle ax, y \rangle = a \langle x, y \rangle.$$

for all $x, y, z \in \mathbb{R}^n$ and $a \in \mathbb{R}$.

b) For $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, show that:

$$||x + ty||^2 = ||x||^2 + 2t \langle x, y \rangle + t^2 ||y||^2 \geq 0 \tag{1}$$

c) By thinking of (1) as a quadratic in t, and considering its possible roots, deduce the *Cauchy-Schwartz* inequality:

$$|\langle x, y \rangle| \leq ||x|| \, ||y|| \,. \tag{2}$$

When does equality hold?

d) Deduce the triangle inequality (I.1).

e) Show the reverse triangle inequality:

$$\big| \, ||x|| - ||y|| \, \big| \leq ||x - y||$$

f) Suppose $x = (x^1, \ldots, x^n)^t \in \mathbb{R}^n$.

i) Show that:

$$\max_{k=1,\ldots,n} \left| x^k \right| \leq ||x|| \,.$$

[Hint: first write $x = x' + x''$ with $x' = (x^1, 0, \ldots, 0)$ and $x'' = (0, x^2, \ldots, x^n)$, and expand]

ii) Show that:

$$||x|| \leq \sqrt{n} \max_{k=1,\ldots,n} \left| x^k \right| .$$

[Hint: write out $||x||^2$ in coordinates and estimate]

Exercise 1.2. Suppose that $(x_i)_{i=0}^{\infty}$ and $(y_i)_{i=0}^{\infty}$ with $x_i, y_i \in \mathbb{R}^n$ are two sequences of vectors with

$$x_i \to x, \quad y_i \to y, \qquad \text{as } i \to \infty.$$

a) Show that

$$x_i + y_i \to x + y \qquad \text{as } i \to \infty.$$

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Questions marked $(*)$ are optional

b) Show that

$$\langle x_i, y_i \rangle \to \langle x, y \rangle \qquad \text{as } i \to \infty,$$

deduce that

$$||x_i|| \to ||x|| \qquad \text{as } i \to \infty.$$

[Hint: Write $\langle x_i, y_i \rangle - \langle x, y \rangle = \langle x_i - x, y_i - y \rangle + \langle x_i - x, y \rangle + \langle x, y_i - y \rangle$ and use the Cauchy-Schwartz inequality (2)]

c) Suppose that $(a_i)_{i=0}^{\infty}$ with $a_i \in \mathbb{R}$ is a sequence of real numbers with $a_i \to a$ as $i \to \infty$. Show that:

$$a_i x_i \to a x \qquad \text{as } i \to \infty.$$

[Hint: Write $a_i x_i - a x = (a_i - a)(x_i - x) + (a_i - a)x + a(x_i - x)$ and use the properties of the norm.]

Exercise 1.3. Which of the following subsets of $\mathbb{R}^n$ is open:

a) $\mathbb{R}^n$?

b) $\emptyset$?

c) $\{x = (x^1, \dots, x^n)^t \in \mathbb{R}^n : x^1 > 0\}$?

d) $\{x = (x^1, \dots, x^n)^t \in \mathbb{R}^n : x^i \in [0, 1)\}$?

e) $\mathbb{Q}^n := \{x = (x^1, \dots, x^n)^t \in \mathbb{R}^n : x^i \in \mathbb{Q}\}$?

Exercise 1.4. Which of the following subsets of $\mathbb{R}^n$ is closed:

a) $\mathbb{R}^n$?

b) $\emptyset$?

c) $\{x = (x^1, \dots, x^n)^t \in \mathbb{R}^n : x^1 \geq 0\}$?

d) $\{x = (x^1, \dots, x^n)^t \in \mathbb{R}^n : x^i \in [0, 1)\}$?

e) $\mathbb{Q}^n := \{x = (x^1, \dots, x^n)^t \in \mathbb{R}^n : x^i \in \mathbb{Q}\}$?

Exercise 1.5. a) Let $(x_i)_{i=0}^{\infty}$ be a sequence of vectors $x_i \in \mathbb{R}^n$ with $x_i \to x$. Suppose that the x_i satisfy $||x_i|| < r$ for all i and some $r > 0$. Show that:

$$||x|| \leq r.$$

[Hint: work by contradiction, assume $||x|| > r$ and show this leads to an absurdity]

b) Show that the closure of the open ball $B_r(y) := \{x \in \mathbb{R}^n : ||x - y|| < r\}$ is the closed ball $\overline{B_r(y)} := \{x \in \mathbb{R}^n : ||x - y|| \leq r\}$.

[Hint: First use part a) to show that the closure is contained in $\overline{B_r(y)}$. Then for each $x \in \overline{B_r(y)}$, find a sequence tending to x.]

Exercise 1.6. Find A°, $\overline{A}$ and ∂A for the following sets:

a) $A = \mathbb{R}^n$.

b) $A = \{0\}$.

c) $A = \{x = (x^1, \ldots, x^n)^t \in \mathbb{R}^n : x^1 \geq 0\}$?

d) $A = \{x = (x^1, \ldots, x^n)^t \in \mathbb{R}^n : x^i \in [0, 1) \; \forall \, i = 1, \ldots, n\}$?

e) $A = \{x = (x^1, \ldots, x^n)^t \in \mathbb{R}^n : x^i \in \mathbb{Q} \; \forall \, i = 1, \ldots, n\}$?

Exercise 1.7. a) Show that if U_1, U_2 are open, then so are:

i) $U_1 \cup U_2$ *ii)* $U_1 \cap U_2$

b) Show that if E_1, E_2 are closed, then so are:

i) $E_1 \cup E_2$ *ii)* $E_1 \cap E_2$

[Hint: use part a) and Theorem I.2]

*c) Suppose $\mathcal{U}$ is any collection of open sets.

i) Show that $\bigcup \mathcal{U}$ is open.

ii) Give an example showing that $\bigcap \mathcal{U}$ need not be open.

iii) What are the analogous statements for closed sets?