Chapter 4

Integration in higher dimensions

4.1 Integration along a curve

4.1.1 Integration of a function along a curve

In many situations, we want to integrate over more interesting shapes than a simple interval. Consider for example the kind of integrals that appear in complex analysis: we want to integrate over some contour in the complex plane. We can in fact do this with the machinery that we've already established, but we first need to make a bit more precise what we mean by a curve.

Definition 4.1. A C^1-curve in $\mathbb{R}^n$ is a differentiable map $\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma' : (a, b) \to \mathbb{R}^n$ given by $\gamma'(p) = D_p\gamma$ is uniformly continuous and non-zero. If $\gamma(a) = \gamma(b)$, we say that the curve is closed.

Example 4.1. The map $\gamma : [0, 2\pi] \to \mathbb{R}^2$ given by:

$$\gamma(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

is a closed C^1-curve.

We can integrate a function defined in the neighbourhood of a C^1-curve as follows. Suppose $\gamma : [a, b] \to \mathbb{R}^n$ is a C^1-curve, that $\Omega \subset \mathbb{R}^n$ with $\gamma([a, b]) \subset \Omega$ and that $f : \Omega \to \mathbb{R}$ is a continuous function. We define the integral of f along γ to be:

$$\int_\gamma f d\ell := \int_a^b f(\gamma(t)) \left|\left|\gamma'(t)\right|\right| dt.$$

Notice that the assumption that f is continuous and γ' is uniformly continuous is stronger than necessary: it is sufficient to assume that $(f \circ \gamma) \times ||\gamma'|| : [a, b] \to \mathbb{R}$ is integrable.

Example 4.2. Take γ to be the curve in Example 4.1 and suppose $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is given by:

$$f(x, y) = \frac{x^2}{x^2 + y^2}$$

Then we calculate

$$f(\gamma(\theta)) = f(\cos\theta, \sin\theta) = \cos^2\theta,$$

and:

$$\gamma'(t) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

so that $||\gamma'(t)|| = 1$ and:

$$\int_\gamma f d\ell = \int_0^{2\pi} cos^2\theta d\theta = \pi.$$

Suppose $\gamma : [a,b] \to \mathbb{R}^n$ is a C^1-curve, that $\Omega \subset \mathbb{R}^n$ with $\gamma([a,b]) \subset \Omega$ and that $f : \Omega \to \mathbb{R}$ is a continuous function. Suppose also that $s : [c,d] \to [a,b]$ satisfies $s(c) = a$, $s(d) = b$ and that $s' : (c,d) \to \mathbb{R}$ is uniformly continuous and positive. Then, by the mean value theorem, s is strictly monotone increasing and continuous, and hence maps $[c,d] \to [a,b]$ bijectively. If we define $\tilde{\gamma} = \gamma \circ s : [c,d] \to \mathbb{R}^n$, then $\tilde{\gamma}$ is a C^1-curve. Moreover,

$$\gamma([a,b]) = \tilde{\gamma}([c,d])$$

so that the curves γ, $\tilde{\gamma}$ coincide as subsets of $\mathbb{R}^n$. We say $\tilde{\gamma}$ is a C^1-*reparameterisation* of γ. By the chain rule, we have for any $t \in (c,d)$ that $\tilde{\gamma}'(t) = s'(t)\gamma'(s(t))$, so that:

$$\left|\left|\tilde{\gamma}'(t)\right|\right| = s'(t)\left|\left|\gamma'(s(t))\right|\right|,$$

using that $s'(t) > 0$. Now, we have:

$$\begin{aligned}\int_{\tilde{\gamma}} f d\ell &= \int_c^d f(\tilde{\gamma}(t))\left|\left|\tilde{\gamma}'(t)\right|\right| dt \\ &= \int_c^d f(\gamma(s(t)))\left|\left|\gamma'(s(t))\right|\right| s'(t)dt \\ &= \int_a^b f(\gamma(s))\left|\left|\gamma'(s)\right|\right| ds = \int_\gamma f d\ell,\end{aligned}$$

by making a change of variables. Thus, the integral along a C^1-curve depends on the curve, rather than the parameterisation. We define the length of a C^1-curve to be:

$$L(\gamma) := \int_\gamma 1 d\ell.$$

By the argument above, this definition of length doesn't depend on the parameterisation of the curve.

Exercise 9.1. Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be given by:

$$\gamma(\theta) = \begin{pmatrix} \theta - \sin\theta \\ 1 - \cos\theta \end{pmatrix}.$$

a) Sketch γ.

b) Find
$$L(\gamma) = \int_\gamma 1 d\ell.$$

This curve is called the *cycloid*, and it is the path traced out by a point on a unit circle which rolls without slipping along the $x-$axis.

4.1.2 Integration of a vector field along a curve

In many physical and mathematical applications, we want to integrate *vectorial* quantities along a curve. For instance, the classical Ampère's law of electromagnetism relates the flux of a stationary current through a surface bounded by a curve to an integral of the magnetic field (a vector) around that curve. We can incorporate this idea into the machinery that we've already developed. Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and that $\gamma : [a, b] \to \Omega$ is a C^1-curve. A *vector field*[1] defined on Ω is simply a map $V : \Omega \to \mathbb{R}^n$, which attaches a vector in $\mathbb{R}^n$ to each point of Ω.

If V is continuous on Ω, we define the integral of V along γ to be:

$$\int_\gamma \langle V, d\boldsymbol{\ell}\rangle = \int_a^b \big\langle V(\gamma(t)), \gamma'(t)\big\rangle\, dt,$$

Notice that since V and γ' are both vectors in $\mathbb{R}^n$, the inner product makes sense at each point $t \in [a, b]$.

As in the case of integration of a function along γ, we would like to show that this quantity doesn't depend on how we parameterise the curve. Let us again suppose that $s : [a, b] \to [c, d]$ satisfies $s(c) = a$, $s(d) = b$ and that $s' : (c, d) \to \mathbb{R}$ is uniformly continuous and positive. Defining $\tilde{\gamma} = \gamma \circ s : [c, d] \to \mathbb{R}^n$, we compute:

$$\begin{aligned}
\int_{\tilde{\gamma}} \langle V, d\boldsymbol{\ell}\rangle &= \int_c^d \big\langle V(\tilde{\gamma}(t)), \tilde{\gamma}'(t)\big\rangle\, dt \\
&= \int_c^d \big\langle V(\gamma(s(t))), \gamma'(s(t))s'(t)\big\rangle\, dt \\
&= \int_c^d \big\langle V(\gamma(s(t))), \gamma'(s(t))\big\rangle\, s'(t)dt \\
&= \int_a^b \big\langle V(\gamma(t)), \gamma'(t)\big\rangle\, dt = \int_\gamma \langle V, d\boldsymbol{\ell}\rangle\,.
\end{aligned}$$

so that the value of the integral of V along γ again depends on the curve, rather than the particular parameterisation that we choose.

Again, the condition that γ is C^1 and V is continuous is stronger than we really require to define the integral of V along γ. It's enough that $\langle V \circ \gamma, \gamma'\rangle : [a, b] \to \mathbb{R}$ is integrable.

[1] This definition is more than adequate for our purposes, but in more exciting situations where Ω is not an open subset of $\mathbb{R}^n$, but a more general manifold the definition of a vector field becomes more subtle. The Manifolds course next year will deal with these very interesting ideas.

Example 4.3. Let us take $\gamma : [0, 2\pi] \to \mathbb{R}^2$ to be given by:

$$\gamma(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

so that γ is the unit circle traversed anticlockwise. We will take as our vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$

$$V : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \end{pmatrix}$$

Now, we calculate:

$$\gamma'(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

and

$$V(\gamma(t)) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

so that $\langle V(\gamma(t)), \gamma'(\theta)\rangle = 1$ and:

$$\int_\gamma \langle V, d\boldsymbol{\ell}\rangle = \int_0^{2\pi} 1dt = 2\pi.$$

4.2 Integration over an area: the Darboux integral on $\mathbb{R}^2$

In this section we are going to consider how to define the integral of a function $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}^2$. The restriction to $\mathbb{R}^2$ is for clarity and to keep the notation sensible. The same basic idea easily extends to integrating over functions of more than two variables. To begin with we will take A to be a rectangular domain: $A = [a_1, b_1] \times [a_2, b_2]$. This is the natural generalisation to two dimensions of the integral over an interval. We will again partition the domain of integration into smaller pieces and approximate the integral above and below by finite sums. This time, rather than representing the area of a set of rectangles, the sums will represent the volume of a set of cuboids, which approximate from above and below the volume below the graph of f. See Figures 4.1, 4.2 for a graphical depiction of this.

4.2.1 Partitions of a square

Suppose that $A = [a_1, b_1] \times [a_2, b_2]$. A partition $\mathcal{P}$ of A is a pair $(\mathcal{P}_1, \mathcal{P}_2)$, where $\mathcal{P}_1 = (x_0, \ldots, x_k)$ is a partition of $[a_1, b_1]$ and $\mathcal{P}_2 = (y_0, \ldots, y_l)$ is a partition of $[a_2, b_2]$. The partition $\mathcal{P}$ divides A into $k \times l$ subdomains:

$$\pi_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}], \qquad i = 0, \ldots, k-1, \quad j = 0, \ldots, l-1.$$

We define $|\pi_{ij}| := (x_{i+1} - x_i) \times (y_{j+1} - y_j)$ to be the area of the square π_{ij}. We also define

$$\text{mesh}\ \mathcal{P} = \max\{\text{mesh}\ \mathcal{P}_1, \text{mesh}\ \mathcal{P}_2\}.$$

We say that $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ refines $\mathcal{P}$ if $\mathcal{P}_1 \preceq \mathcal{Q}_1$ *and* $\mathcal{P}_2 \preceq \mathcal{Q}_2$. In this case, each subdomain of $\mathcal{P}$ is split into a finite number of subdomains of $\mathcal{Q}$. An example of a

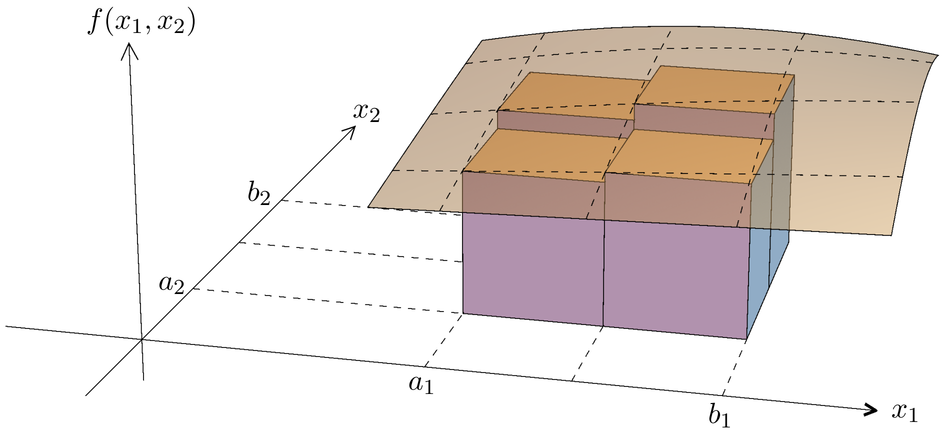

Figure 4.1 Approximating the volume beneath a graph over the plane

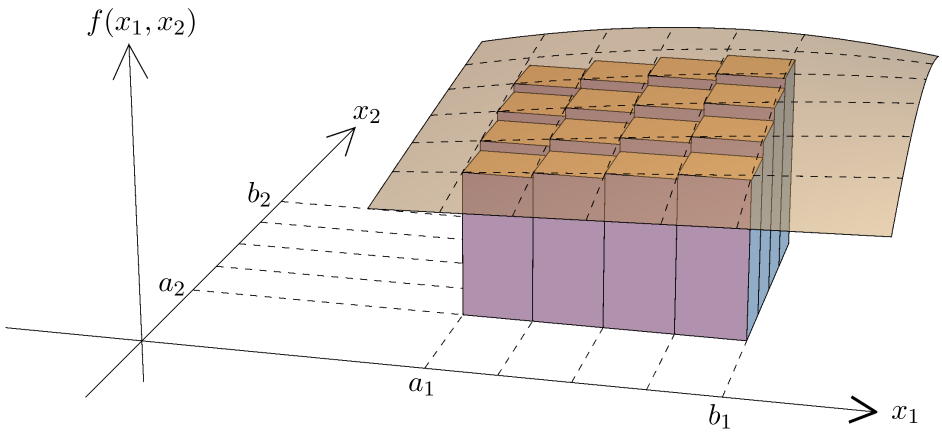

Figure 4.2 Approximating with a finer partition

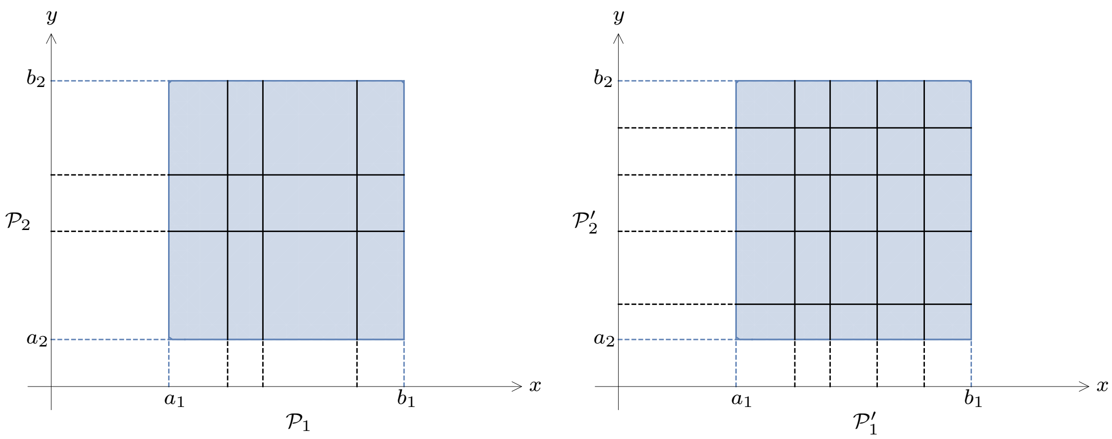

Figure 4.3 Two partitions $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$ and $\mathcal{P}' = (\mathcal{P}'_1, \mathcal{P}'_2)$ of the domain A, with $\mathcal{P} \preceq \mathcal{P}'$.

partition and a refinement of that partition are shown in Figure 4.3. If $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$ and $\mathcal{P}' = (\mathcal{P}'_1, \mathcal{P}'_2)$ are any two partitions, we define their common refinement to be $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$, where $\mathcal{R}_1$ is the common refinement of $\mathcal{P}_1, \mathcal{P}'_1$ and $\mathcal{R}_2$ is the common refinement of $\mathcal{P}_2, \mathcal{P}'_2$.

4.2.2 Upper and Lower Darboux Sums

We now consider a bounded function $f : A \to \mathbb{R}$. For each $i = 0, \ldots, k-1$ and $j = 0, \ldots, l-1$, we define:

$$m_{ij} := \inf_{x \in \pi_{ij}} f(x),$$
$$M_{ij} := \sup_{x \in \pi_{ij}} f(x),$$

which are respectively the greatest lower and least upper bounds for f in the subdomain π_{ij}. As we've assumed f to be bounded these are well defined. We then define the lower and upper sums of f associated to the partition $\mathcal{P}$ is the obvious way:

$$L(f, \mathcal{P}) := \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} m_{ij} \, |\pi_{ij}|$$
$$U(f, \mathcal{P}) := \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} m_{ij} \, |\pi_{ij}| \, .$$

Abusing notation slightly, we'll sometimes write $\pi \in \mathcal{P}$ to mean π is a subdomain of $\mathcal{P}$. The allows us to write the sums more compactly as:

$$L(f, \mathcal{P}) = \sum_{\pi \in \mathcal{P}} \inf_{x \in \pi} f(x) \, |\pi|$$
$$U(f, \mathcal{P}) = \sum_{\pi \in \mathcal{P}} \sup_{x \in \pi} f(x) \, |\pi| \, .$$

where by the sum over $\pi \in \mathcal{P}$ we mean summing over all distinct subdomains of $\mathcal{P}$. As in the one-dimensional case, since $m_{ij} \leq M_{ij}$, we certainly have:

$$L(f, \mathcal{P}) \leq U(f, \mathcal{P}).$$

In fact, almost all of the results that we proved for upper and lower sums in the one-dimensional case carry over to the higher dimensional setting. In particular, we have:

Theorem 4.1. *Let $A = [a_1, b_1] \times [a_2, b_2]$ and $f : A \to \mathbb{R}$ be bounded. Suppose that $\mathcal{P}$, $\mathcal{Q}$ are partitions of A with $\mathcal{Q}$ a refinement of $\mathcal{P}$. Then:*

$$L(f, \mathcal{P}) \leq L(f, \mathcal{Q}) \leq U(f, \mathcal{Q}) \leq U(f, \mathcal{P}).$$

Proof. Write $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$ and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$. Suppose first that $\mathcal{P}_1 = (x_0, \ldots, x_k)$, $\mathcal{Q}_1 = (x_0, \ldots, x_l, c, x_{l+1}, \ldots x_k)$, and $\mathcal{P}_2 = \mathcal{Q}_2 = (y_0, \ldots, y_m)$, so that $\mathcal{P}$ and $\mathcal{Q}$ differ only in that a single subinterval $[x_l, x_{l+1}]$ in the $x-$direction has been split into $[x_l, c]$ and $[c, x_{l+1}]$, with the partition in the $y-$direction left alone. From basic properties of the infimum, we have:

$$\inf_{x \in [x_l, c] \times [y_j, y_{j+1}]} f(x) \geq \inf_{x \in [x_l, x_{l+1}] \times [y_j, y_{j+1}]} f(x),$$
$$\inf_{x \in [c, x_{l+1}] \times [y_j, y_{j+1}]} f(x) \geq \inf_{x \in [x_l, x_{l+1}] \times [y_j, y_{j+1}]} f(x),$$

for any $y = 0, \ldots, m-1$. Now, if we take the difference of the lower sums associated to $\mathcal{P}$ and $\mathcal{Q}$, then the only terms which will not cancel will be those involving the subinterval $[x_l, x_{l+1}]$, see Figure 4.4:

$$\begin{aligned} L(f, \mathcal{Q}) - L(f, \mathcal{P}) &= \sum_{j=0}^{m-1} (c - x_l)(y_{j+1} - y_j) \inf_{x \in [x_l, c] \times [y_j, y_{j+1}]} f(x) \\ &\quad + \sum_{j=0}^{m-1} (x_{l+1} - c)(y_{j+1} - y_j) \inf_{x \in [c, x_{l+1}] \times [y_j, y_{j+1}]} f(x) \\ &\quad - \sum_{j=0}^{m-1} (x_{l+1} - x_l)(y_{j+1} - y_j) \inf_{x \in [x_l, x_{l+1}] \times [y_j, y_{j+1}]} f(x) \\ &\geq \sum_{j=0}^{m-1} [(c - x_l) + (x_{l+1} - c) - (x_{l+1} - x_l)] \\ &\qquad \times (y_{j+1} - y_j) \times \inf_{x \in [x_l, x_{l+1}] \times [y_j, y_{j+1}]} f(x), \\ &= 0. \end{aligned}$$

Thus we have:

$$L(f, \mathcal{P}) \leq L(f, \mathcal{Q}).$$

Now, note that by Exercise 9.2 *a*), we have $U(f, \mathcal{P}) = -L(-f, \mathcal{P})$, so that:

$$U(f, \mathcal{Q}) = -L(-f, \mathcal{Q}) \leq -L(-f, \mathcal{P}) = U(f, \mathcal{P}),$$

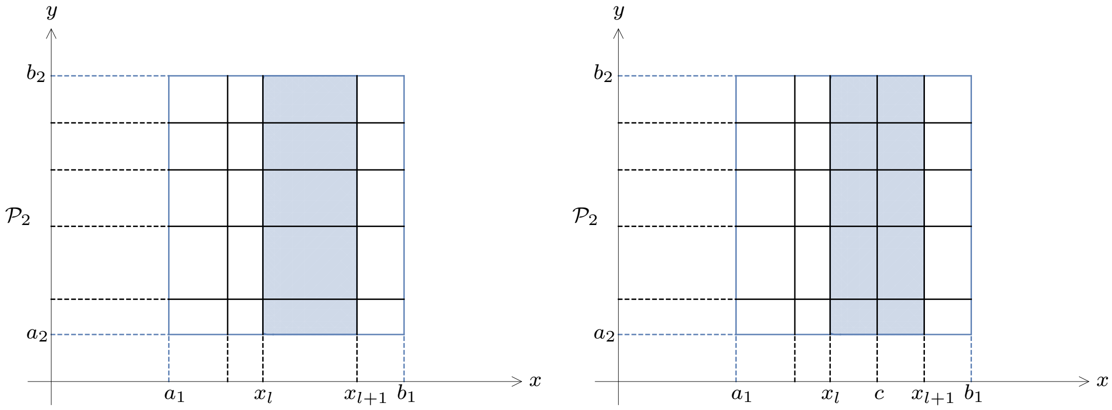

Figure 4.4 The partitions $\mathcal{P}$, $\mathcal{Q}$ from the proof of Theorem 4.1. The subdomains giving rise to terms in the upper and lower sums which do not cancel are shaded.

so that we have established

$$L(f, \mathcal{P}) \leq L(f, \mathcal{Q}) \leq U(f, \mathcal{Q}) \leq U(f, \mathcal{P}).$$

in the special case where $\mathcal{Q}$ is obtained from $\mathcal{P}$ by adding one point in the $x-$partition. An identical argument establishes the result in the case where $\mathcal{P}_1 = \mathcal{Q}_1$ and $\mathcal{P}_2$, $\mathcal{Q}_2$ differ only by one point. For the general result, we observe that if $\mathcal{P} \preceq \mathcal{Q}$ then we can obtain $\mathcal{Q}$ by inserting a finite number of points into $\mathcal{P}$, and we can apply our special case result at each stage to obtain the general result. □

Corollary 4.2. *Let $A = [a_1, b_1] \times [a_2, b_2]$ and let $f : A \to \mathbb{R}$ be bounded. Suppose $\mathcal{P}, \mathcal{P}'$ are any two partitions of A. Then:*

$$L(f, \mathcal{P}) \leq U(f, \mathcal{P}').$$

Proof. Let $\mathcal{Q}$ be the common refinement of $\mathcal{P}, \mathcal{P}'$. Then by Theorem 4.1 we have:

$$L(f, \mathcal{P}) \leq L(f, \mathcal{Q}) \leq U(f, \mathcal{Q}) \leq U(f, \mathcal{P}'),$$

which gives the result. □

Exercise 9.2. Let $A = [a_1, b_1] \times [a_2, b_2]$ be a square domain in $\mathbb{R}^2$ and let $\mathcal{P}$ be any partition of A. Suppose $f, g : A \to \mathbb{R}$ are bounded functions, and that $\lambda \geq 0$. Show that:

a) $U(-f, \mathcal{P}) = -L(f, \mathcal{P})$,

b) $L(-f, \mathcal{P}) = -U(f, \mathcal{P})$,

c) $U(\lambda f, \mathcal{P}) = \lambda U(f, \mathcal{P})$,

d) $L(\lambda f, \mathcal{P}) = \lambda L(f, \mathcal{P})$,

e) $U(f + g, \mathcal{P}) \leq U(f, \mathcal{P}) + U(g, \mathcal{P})$,

f) $L(f + g, \mathcal{P}) \geq L(f, \mathcal{P}) + L(g, \mathcal{P})$.

4.2.3 The Darboux integral

We are now ready to define the integral:

Definition 4.2. The *lower* and *upper Darboux integrals* are defined as follows:

$$\underline{\int_A} f(x)dx := \sup_{\mathcal{P}} L(f, \mathcal{P}) \qquad \overline{\int_A} f(x)dx := \inf_{\mathcal{P}} U(f, \mathcal{P})$$

where the supremum and infimum are taken over all partitions of $[a, b]$. By Corollary 4.2 we have that:

$$\underline{\int_A} f(x)dx \leq \overline{\int_A} f(x)dx.$$

We say that a bounded function $f : [a, b] \to \mathbb{R}$ is *Darboux integrable*, or simply *integrable*, if it is bounded, and the upper and lower Darboux integrals are equal. In this case, we write:

$$\int_A f(x)dx := \underline{\int_A} f(x)dx = \overline{\int_A} f(x)dx.$$

Notice that the lower and upper Darboux integrals are always well defined for a bounded function f.

As a notational convention, x appears in the integral as a 'dummy variable', much as in the one-dimensional case. Sometimes to emphasise that this is a two dimensional integral, it is written:

$$\int_A f(x)d^2x, \quad \text{or} \quad \int_A f(x)dx^1dx^2.$$

The results that we established in Chapter 2 concerning integrable functions defined on the interval carry over wholesale to the two dimensional setting. We'll state them again, but only repeat the proofs where some modification is necessary. You should look back at the proofs from Chapter 2 and check that you understand how they extend to the two dimensional case.

Theorem 4.3 (Linearity of the integral)**.** *Let $A = [a_1, b_1] \times [a_2, b_2]$. Suppose that $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are both integrable, and that $\lambda \in \mathbb{R}$. Then $f + \lambda g$ is integrable and we have:*

$$\int_A (f(x) + \lambda g(x))\, dx = \int_A f(x)dx + \lambda \int_A g(x)dx.$$

Proof. Repeat proof of Theorem 2.4 □

Lemma 4.4. *Let $A = [a_1, b_1] \times [a_2, b_2]$. A function $f : A \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$ there exists a partition $\mathcal{P}$ such that:*

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon. \tag{4.1}$$

Proof. Repeat proof of Lemma 2.5 □

Theorem 4.5 (Additivity of the integral)**.** *Let $A = [a_1, b_1] \times [a_2, b_2]$. Suppose $c \in (a_1, b_1)$ and set $A_L = [a_1, c] \times [a_2, b_2], A_R = [c, b_1] \times [a_2, b_2]$. Then $f : A \to \mathbb{R}$ is integrable if and only if $f|_{A_L}$ and $f|_{A_R}$ are integrable, and:*

$$\int_A f(x)dx = \int_{A_L} f(x)dx + \int_{A_R} f(x)dx.$$

Proof. Repeat proof of Theorem 2.6 □

Theorem 4.6. *Let $A = [a_1, b_1] \times [a_2, b_2]$. Suppose $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are both integrable. Then so is:*

i) $|f|$,

ii) f^2,

iii) fg.

Proof. Repeat proof of Theorem 2.8 □

Theorem 4.7. *Let $A = [a_1, b_1] \times [a_2, b_2]$. Suppose that $f : A \to \mathbb{R}$ is continuous on A. Then f is integrable.*

Proof. This time we need to slightly modify the proof. Let $|A| := (b_1 - a_1) \times (b_2 - a_2)$. Fix $\epsilon > 0$. Since f is continuous on A, it is uniformly continuous. Thus there exists $\delta > 0$ such that for all $x, y \in A$ with $||x - y|| < \delta$ we have $|f(x) - f(y)| < \frac{1}{2|A|}\epsilon$. Pick a partition $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$ with $\mathcal{P}_1 = (x_0, \ldots, x_k)$ and $\mathcal{P}_2 = (y_0, \ldots, y_l)$ such that:

$$\sqrt{(x_{i+1} - x_i)^2 + (y_{j+1} - y_j)^2} < \delta$$

for all $i = 0, \ldots k - 1, j = 0, \ldots, l - 1$. We can do this by (for example) setting:

$$\begin{aligned} x_i &= a_1 + \frac{b_1 - a_1}{k} i, \qquad i = 0, \ldots, k, \\ y_j &= a_2 + \frac{b_2 - a_2}{l} j, \qquad j = 0, \ldots, l, \end{aligned}$$

where k, l are chosen sufficiently large that

$$(b_1 - a_1)\sqrt{2} < k\delta, \qquad (b_2 - a_2)\sqrt{2} < l\delta.$$

Now, by construction, for $x, y \in \pi_{ij}$ we have:

$$|f(x) - f(y)| < \frac{1}{2|A|}\epsilon,$$

which implies that:

$$\sup_{x \in \pi_{ij}} f(x) - \inf_{x \in \pi_{ij}} f(x) \leq \frac{1}{2|A|}\epsilon < \frac{1}{|A|}\epsilon.$$

Now, we can estimate:

$$\begin{aligned} U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\left[\sup_{x\in\pi_{ij}} f(x) - \inf_{x\in\pi_{ij}} f(x)\right]|\pi_{ij}| \\ &< \sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\frac{\epsilon}{|A|}|\pi_{ij}| \\ &= \frac{\epsilon}{|A|}\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}(x_{i+1}-x_i)\times(y_{j+1}-y_j) \\ &= \frac{\epsilon}{|A|}\sum_{i=0}^{k-1}(x_{i+1}-x_i)\times\sum_{j=0}^{l-1}(y_{j+1}-y_j) \\ &= \epsilon, \end{aligned}$$

as we have a pair of telescoping sums. Thus for any $\epsilon > 0$, we have found a partition $\mathcal{P}$ such that:

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon,$$

which implies f is integrable. □

Again, notice that it's enough that A is uniformly continuous on $(a_1, b_1) \times (a_2, b_2)$ for this argument to work. Finally, we will state without proof the following result concerning Riemann sums. A tagged partition is a partition $\mathcal{P}$ together with a choice of a point in each subdomain $\tau = \{t_\pi | \pi \in \mathcal{P}\}$. The Riemann sum corresponding to the tagged partition $(\mathcal{P}, \tau)$ is:

$$R(f,\mathcal{P},\tau) = \sum_{\pi\in\mathcal{P}} f(t_\pi)\,|\pi|\,.$$

Theorem 4.8. *Let $A = [a_1, b_1] \times [a_2, b_2]$ and let $f : [a, b] \to \mathbb{R}$ be an integrable function. Suppose $((\mathcal{P}_l, \tau_l))_{l=0}^{\infty}$ is a sequence of tagged partitions with mesh $\mathcal{P}_l \to 0$. Then we have:*

$$R(f,\mathcal{P}_l,\tau_l) \to \int_A f(x)dx.$$

Exercise 9.3. Let $A = [-1, 1] \times [-1, 1]$. Show that the function $f : A \to \mathbb{R}$ given by:

$$f(x,y) = \begin{cases} 1 & x \notin \mathbb{Q}, \text{ and } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

is integrable, and:

$$\int_A f(z)dz = 0.$$

Exercise 9.4. Let $A = [a_1, b_1] \times [a_2, b_2]$ and set $|A| = (b_1 - a_1) \times (b_2 - a_2)$. Suppose $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ are integrable.

a) Show that:

$$|A| \inf_{x \in A} f(x) \leq \int_A f(x)dx \leq |A| \sup_{x \in A} f(x).$$

b) Establish the estimate:

$$\left| \int_A f(x)dx \right| \leq |A| \sup_{x \in A} |f(x)|$$

c) Show that if $0 \leq f(x)$ for all $x \in A$ then:

$$0 \leq \int_A f(x)dx$$

d) Show that if $f(x) \leq g(x)$ for all $x \in A$ then:

$$\int_A f(x)dx \leq \int_A g(x)dx$$

e) Prove that:

$$\left| \int_A f(x)dx \right| \leq \int_A |f(x)| \, dx$$

Exercise 9.5. Let $A = [a_1, b_1] \times [a_2, b_2]$. Suppose that $f : A \to \mathbb{R}$ is continuous. Show that the map $F : [a_2, b_2] \to \mathbb{R}$ given by:

$$F(y) = \int_{a_1}^{b_1} f(x, y)dx$$

is continuous.

4.2.4 Relation to repeated integrals

When it comes to evaluating an integral over a square domain in practice, it is useful to connect the integral to repeated one-dimensional integrals. Roughly speaking, once can imagine first sending the mesh of the partition in the $x-$direction to zero, and then the mesh of the partition in the $y-$direction. More concretely, if $A = [a_1, b_1] \times [a_2, b_2]$ and $f : A \to \mathbb{R}$ is continuous, for each $y \in [a_2, b_2]$ we can consider the function:

$$\begin{array}{cccc} g_y & : & [a_1, b_1] & \to & \mathbb{R} \\ & & x & \mapsto & f(x, y). \end{array}$$

For each y this will be a continuous function, so we can define:

$$G_y = \int_{a_1}^{b_1} g_y(x)dx = \int_{a_1}^{b_1} f(x, y)dx$$

By Exercise 9.5 we have that the map $G : y \mapsto G_y$ is continuous, and so it makes sense to compute:

$$I_{12} := \int_{a_2}^{b_2} G(y)dy = \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x, y)dx \right) dy.$$

Similarly, we could first integrate over y to find:

$$I_{21} := \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x, y) dy \right) dx$$

One might expect that these double integrals are related to

$$I = \int_A f(z) dz,$$

and indeed we shall show:

$$I_{12} = I_{21} = I.$$

For a function $f : A \to \mathbb{R}$ which is merely integrable (rather than continuous) we have the problem that the integral over x may not be defined at all values of y (and vice versa). For example, the function in Exercise 9.3 is integrable over $[-1, 1] \times [-1, 1]$, however:

$$x \mapsto f(x, 0)$$

is not integrable, so the expression:

$$\int_{-1}^{1} \left(\int_{-1}^{1} f(x, y) dx \right) dy$$

does not make sense. We can fix this by replacing the inner integral with an upper (or lower) integral.

Theorem 4.9 (Fubini's theorem)**.** *Let* $A = [a_1, b_1] \times [a_2, b_2]$ *and suppose* $f : A \to \mathbb{R}$ *is integrable. Then the two maps* $\mathcal{U}, \mathcal{L} : [a_2, b_2] \to \mathbb{R}$ *given by:*

$$\mathcal{U} : y \mapsto \overline{\int_{a_1}^{b_1}} f(x, y) dx,$$
$$\mathcal{L} : y \mapsto \underline{\int_{a_1}^{b_1}} f(x, y) dx,$$

are both integrable, and:

$$\int_A f(z) dz = \int_{a_2}^{b_2} \left(\underline{\int_{a_1}^{b_1}} f(x, y) dx \right) dy = \int_{a_2}^{b_2} \left(\overline{\int_{a_1}^{b_1}} f(x, y) dx \right) dy.$$

Proof (*). Suppose $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$ be a partition of A with $\mathcal{P}_1 = (x_0, \ldots, x_k)$ and $\mathcal{P}_2 = (y_0, \ldots, y_l)$.

1. We first estimate:

$$\begin{aligned} L(f, \mathcal{P}) &= \sum_{\pi \in \mathcal{P}} \inf_{z \in \pi} f(z) \, |\pi| \\ &= \sum_{j=0}^{l-1} \left(\sum_{i=0}^{k-1} \inf_{z \in \pi_{ij}} f(z)(x_{i+1} - x_i) \right) (y_{j+1} - y_j) \end{aligned}$$

Note that if $\pi_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ then:

$$\inf_{z\in\pi_{ij}} f(z) \leq \inf_{x\in[x_{i+1},x_i]} f(x,y), \qquad \text{for all } y \in [y_{j+1}, y_j].$$

So that:

$$\begin{aligned}\sum_{i=0}^{k-1} \inf_{z\in\pi_{ij}} f(z)(x_{i+1}-x_i) &\leq \sum_{i=0}^{k-1} \inf_{x\in[x_{i+1},x_i]} f(x,y)(x_{i+1}-x_i)\\ &= L(g_y, \mathcal{P}_1)\\ &\leq \underline{\int_{a_1}^{b_1}} f(x,y)dx = \mathcal{L}(y).\end{aligned}$$

for all $y \in [y_{j+1}, y_j]$. Taking the infimum over $y \in [y_{j+1}, y_j]$, we have:

$$\sum_{i=0}^{k-1} \inf_{z\in\pi_{ij}} f(z)(x_{i+1}-x_i) \leq \inf_{y\in[y_{j+1},y_j]} \mathcal{L}(y).$$

We deduce that:

$$L(f,\mathcal{P}) \leq \sum_{j=0}^{l-1} \inf_{y\in[y_{j+1},y_j]} \mathcal{L}(y)(y_{j+1}-y_j) = L(\mathcal{L}, \mathcal{P}_2).$$

2. Next we estimate:

$$\begin{aligned}U(f,\mathcal{P}) &= \sum_{\pi\in\mathcal{P}} \sup_{z\in\pi} f(z)\,|\pi|\\ &= \sum_{j=0}^{l-1}\left(\sum_{i=0}^{k-1} \sup_{z\in\pi_{ij}} f(z)(x_{i+1}-x_i)\right)(y_{j+1}-y_j)\end{aligned}$$

Note that if $\pi_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ then:

$$\sup_{z\in\pi_{ij}} f(z) \geq \sup_{x\in[x_{i+1},x_i]} f(x,y), \qquad \text{for all } y \in [y_{j+1}, y_j].$$

So that:

$$\begin{aligned}\sum_{i=0}^{k-1} \sup_{z\in\pi_{ij}} f(z)(x_{i+1}-x_i) &\geq \sum_{i=0}^{k-1} \sup_{x\in[x_{i+1},x_i]} f(x,y)(x_{i+1}-x_i)\\ &= U(g_y, \mathcal{P}_1)\\ &\geq \overline{\int_{a_1}^{b_1}} f(x,y)dx = \mathcal{U}(y).\end{aligned}$$

for all $y \in [y_{j+1}, y_j]$. Taking the supremum over $y \in [y_{j+1}, y_j]$, we have:

$$\sum_{i=0}^{k-1} \sup_{z \in \pi_{ij}} f(z)(x_{i+1} - x_i) \geq \sup_{y \in [y_{j+1}, y_j]} \mathcal{U}(y).$$

We deduce that:

$$U(f, \mathcal{P}) \geq \sum_{j=0}^{l-1} \sup_{y \in [y_{j+1}, y_j]} \mathcal{U}(y)(y_{j+1} - y_j) = U(\mathcal{U}, \mathcal{P}_2).$$

3. Recalling that $\mathcal{L}(y) \leq \mathcal{U}(y)$ for all $y \in [a_2, b_2]$ we have:

$$L(f, \mathcal{P}) \leq L(\mathcal{L}, \mathcal{P}_2) \leq U(\mathcal{L}, \mathcal{P}_2) \leq U(\mathcal{U}, \mathcal{P}_2) \leq U(f, \mathcal{P}).$$

Since f is integrable, we have:

$$\inf_{\mathcal{P}} U(f, \mathcal{P}) = \sup_{\mathcal{P}} L(f, \mathcal{P}) = \int_A f(z) dz.$$

We deduce that:

$$\inf_{\mathcal{P}_2} U(\mathcal{L}, \mathcal{P}_2) = \sup_{\mathcal{P}_2} L(\mathcal{L}, \mathcal{P}_2) = \int_A f(z) dz,$$

so that $\mathcal{L}$ is integrable and:

$$\int_A f(z) dz = \int_{a_2}^{b_2} \left(\underline{\int_{a_1}^{b_1}} f(x, y) dx \right) dy.$$

The equivalent statement for $\mathcal{U}$ follows from the inequalitites:

$$L(f, \mathcal{P}) \leq L(\mathcal{L}, \mathcal{P}_2) \leq L(\mathcal{U}, \mathcal{P}_2) \leq U(\mathcal{U}, \mathcal{P}_2) \leq U(f, \mathcal{P}).$$ □

Obviously, there was nothing special about our decision to integrate first over the $x-$coordinate and then the y. We could equally have established that:

$$\int_A f(z) dz = \int_{a_1}^{b_1} \left(\underline{\int_{a_2}^{b_2}} f(x, y) dy \right) dx = \int_{a_1}^{b_1} \left(\overline{\int_{a_2}^{b_2}} f(x, y) dy \right) dx.$$

In the case where f is continuous, we have the immediate corollary:

Corollary 4.10. *Suppose $f : A \to \mathbb{R}$ is continuous. Then:*

$$\int_A f(z) dz = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x, y) dy \right) dx = \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x, y) dx \right) dy.$$

Proof. Since f is continuous, so is the map $g_y : x \mapsto f(x, y)$ for each $y \in [a_2, b_2]$, thus g_y is integrable and we have:

$$\underline{\int_{a_1}^{b_1}} f(x, y) dx = \int_{a_1}^{b_1} f(x, y) dx,$$

and equivalently for the integrals keeping x fixed. □

4.2.5 Non-square domains

Thus far we have restricted our attention to functions define on a rectangle $A = [a_1, b_1] \times [a_2, b_2]$. Of course, we often wish to integrate over more exotic shapes (circles, ellipses, stars, etc.). Recall the characteristic function of a set $C \subset \mathbb{R}^2$ is the function $\chi_C : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$\chi_C(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C, \end{cases}$$

If $C \subset A$ for some closed rectangle and $f : A \to \mathbb{R}$ is bounded, then:

$$\int_C f(x)dx = \int_A \chi_C(x)f(x)dx,$$

provided $\chi_C(x)f(x)$ is integrable. By Theorem 4.6 this will certainly be the case if χ_C and f are separately integrable.

Definition 4.3. A bounded set C is called *Jordan measurable* if χ_C is integrable. For such a set we can define the *Jordan measure* to be:

$$m(C) = \int_C 1dx = \int_A \chi_C(x)dx,$$

where A is any rectangle enclosing C.

The Jordan measure is a notion of area for a reasonably large class of sets. Not every set is Jordan measurable. For example, if we take

$$C = \{(x, y)^t \big| -1 < x, y < 1, \quad x, y \in \mathbb{Q}\}$$

then by a similar argument to Example 2.3 we can show that χ_C is not integrable. In fact, in the next section we shall see that the criterion that a set be Jordan measurable is that the boundary be 'small' in some appropriate sense.

A word of caution. Although the nomenclature 'Jordan measure' is reasonably established, one should be careful because the word "measure" has a technical meaning which you will be introduced to in the measure theory course, however the Jordan measure is not a measure in this sense.

4.2.6 (*) Lebesgue's criterion for integrability

Sets of measure zero

A domain π is a closed rectangle if it is of the form $\pi = [a_1, b_1] \times [a_2, b_2]$ and if so, then $|\pi| = (b_1 - a_1) \times (b_2 - a_2)$. A set $A \subset \mathbb{R}^2$ is said to have *measure zero* if the following property holds: given $\epsilon > 0$, we can find a countable collection of rectangular domains $\{\pi_i\}_{i=1}^{\infty}$ such that:

$$A \subset \bigcup_{i=1}^{\infty} \pi_i,$$

and:

$$\sum_{i=1}^{\infty} |\pi_i| < \epsilon.$$

A set has measure zero if it can be completely contained in a set of arbitrarily small area.

Example 4.4. Any countable set $A \subset \mathbb{R}^2$ has measure zero. To see this, we write $A = \{x_1, x_2, \ldots\}$ with $x_i = (x_i^1, x_i^2)^t \in \mathbb{R}^2$. Let us then define:

$$\pi_i = [x_i^1 - \epsilon 2^{-\frac{i}{2}-2}, x_i^1 + \epsilon 2^{-\frac{i}{2}-2}] \times [x_i^2 - \epsilon 2^{-\frac{i}{2}-1}, x_i^2 + \epsilon 2^{-\frac{i}{2}-1}]$$

We have that:

$$|\pi_i| = \epsilon 2^{-i-1}$$

so that:

$$\sum_{i=1}^{\infty} |\pi_i| = \frac{\epsilon}{2} < \epsilon.$$

In fact, we can make a more general statement

Theorem 4.11. *Suppose that $A_1, A_2, \ldots$ is a countable collection of sets of measure zero. Then*

$$A = \bigcup_{j=1}^{\infty} A_j$$

has measure zero.

Proof. Fix $\epsilon > 0$. Since each A_j has measure zero, we can find a countable collection $\mathcal{U}_j = \{\pi_{j,i}\}_{i=1}^{\infty}$ of closed rectangles such that:

$$A_j \subset \bigcup_{i=1}^{\infty} \pi_{j,i},$$

and

$$\sum_{i=1}^{\infty} |\pi_{j,i}| < \epsilon 2^{-j-1}.$$

Now, the set:

$$\mathcal{U} = \bigcup_{j=1}^{\infty} \mathcal{U}_j$$

is a countable union of countable sets, so it is countable. We can therefore write $\mathcal{U} = \{\Pi_k\}_{k=1}^{\infty}$. We certainly have that:

$$A \subset \bigcup_{k=1}^{\infty} \Pi_k,$$

and moreover:

$$\sum_{k=1}^{\infty} |\Pi_k| = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\pi_{j,i}| = \sum_{j=1}^{\infty} \epsilon 2^{-j-1} = \frac{\epsilon}{2} < \epsilon.$$

□

Theorem 4.12. *Suppose $\gamma : [a, b] \to \mathbb{R}^2$ is a C^1-curve. Then $\gamma([a, b])$ has measure zero.*

Proof. We write:

$$\gamma(s) = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$$

for $x, y : [a, b] \to \mathbb{R}$. First, note that since $\gamma'(a, b) \to \mathbb{R}^2$ is uniformly continuous, so are $x', y' : (a, b) \to \mathbb{R}$ and so x', y' extend continuously to $[a, b]$ by Exercise 3.6. By the extreme value theorem, we conclude that both x' and y' are bounded, so that there exists K with:

$$\left|x'(s)\right| \leq K, \qquad \left|y'(s)\right| \leq K.$$

By Exercise 2.6 we deduce that if $s_1, s_2 \in [a, b]$, then:

$$|x(s_1) - x(s_2)| \leq K\,|s_1 - s_2|\,, \qquad |y(s_1) - y(s_2)| \leq K\,|s_1 - s_2|$$

Fix $\epsilon > 0$. Let us define for $n \in \mathbb{N}^*$:

$$s_i = a + (b - a)\frac{i}{n}, \qquad i = 0, \ldots, n.$$

We define the rectangles:

$$\pi_i = [x(s_i) - k, x(s_i) + k] \times [y(s_i) - k, y(s_i) + k],$$

where:

$$k = K\frac{(b - a)}{n}.$$

Now, for any $s \in [a, b]$, there is an s_i with $|s - s_i| < \frac{(b-a)}{n}$. If this is the case, we know that:

$$|x(s) - x(s_i)| \leq K\frac{(b - a)}{n}, \qquad |y(s) - y(s_i)| \leq K\frac{(b - a)}{n},$$

which implies that $\gamma(s) \in \pi_i$. Thus we have:

$$\gamma([a, b]) \subset \bigcup_{i=0}^{n} \pi_i.$$

On the other hand,

$$|\pi_i| = 4K^2\frac{(b - a)^2}{n^2}$$

so that:

$$\sum_{i=0}^{n} |\pi_i| = 4K^2\frac{(b - a)^2}{n^2}(n + 1).$$

By taking n sufficiently large, we can ensure that the right hand side of this expression is as small as we like, in particular less than ϵ, and so we have exhibited a collection of rectangles which contains $\gamma([a, b])$ and has total area less than ϵ. □

Lebesgue's criterion

We shall state without proof a very useful characterisation of the integrable functions.

Theorem 4.13 (Lebesgue's criterion for integrability)**.** *Let $A = [a_1, b_1] \times [a_2, b_2]$ and let $f : A \to \mathbb{R}$ be a bounded function. Then f is integrable if, and only if, the set:*

$$\{p \in A \big| f \text{ is not continuous at } p\}$$

has measure zero.

Those students interested in seeing the proof of this result can find it in the book "Calculus on Manifolds" by Spivak, Theorem 3-8.

A very useful consequence of this result is the following:

Corollary 4.14. *A set $B \subset A = [a_1, b_1] \times [a_2, b_2]$ is Jordan measurable if, and only if, ∂B has measure zero.*

Proof. We need to show that $\chi_B : A \to \mathbb{R}$ is integrable. By Lebesgue's criterion, this is true if and only if the set of points at which χ_B is discontinuous has measure zero. Thus what we need to show is that χ_B is not continuous at p if and only if $p \in \partial B$.

First, suppose that $p \in B^\circ$. Then there exists $\delta > 0$ such that $B_\delta(p) \in B$, and thus $\chi_B(q) = 1$ for any q with $||q - p|| < \delta$. For such q, we have:

$$|\chi_B(q) - \chi_B(p)| = 0,$$

so χ_B is continuous at p.

Next, suppose that $p \in \overline{B}^c$. Since the complement of a closed set is open, there exists $\delta > 0$ such that $B_\delta(p) \subset \overline{B}^c$. Thus if $|q - p| < \delta$ we have $\chi_B(p) = \chi_B(q) = 0$ and again χ_B is continuous at p. Thus χ_B is continuous on $\overline{B}^c$ and B°. It remains to show that χ_B is discontinuous on $\partial B = \overline{B} \setminus B^\circ$.

Finally, suppose that $p \in \partial B$. Then since $p \notin B^\circ$, for any $\delta > 0$ we have that $B_\delta(p) \cap B^c \neq \emptyset$. In particular, for any $i \in \mathbb{N}$ we can find q_i with $q_i \notin B$ and $||q_i - p|| < 2^{-i}$. Thus $q_i \to p$ as $n \to \infty$. We have:

$$\chi_B(q_i) = 0 \to 0,$$

as $n \to \infty$. On the other hand, since $p \in \overline{B}$, there exists a sequence $(q_i)_{i=0}^\infty$ with $q_i' \in B$ and $q_i' \to p$. For this sequence we have:

$$\chi_B(q_i') = 1 \to 1,$$

which implies that χ_B is not continuous at p. □

With this result, we are in a position to show that many 'reasonable' domains are Jordan integrable. For example any domain whose boundary can be (piecewise) parameterised by a C^1−curve will necessarily be Jordan integrable.

4.2.7 (*) Generalisation to $\mathbb{R}^n$

It is clear that the same basic approach we've described above can be generalised to integrate over a hypercube $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. The only real challenge is in handling the notation, which gets very cumbersome very quickly. We define a partition of A to be a $n-$tuple of partitions $\mathcal{P} = (\mathcal{P}^1, \ldots, \mathcal{P}^n)$, where $\mathcal{P}^i = (x^i_0, \ldots, x^i_{k_i})$ is a partition of the interval $[a_i, b_i]$. The subdomains of $\mathcal{P}$ are the hypercubes of the form:

$$\pi_{i_1,\ldots,i_n} = [x^1_{i_1}, x^1_{i_1+1}] \times [x^2_{i_2}, x^2_{i_2+1}] \times \cdots \times [x^n_{i_n}, x^n_{i_n+1}],$$

where $0 \leq i_j \leq k_i - 1$. We define

$$|\pi_{i_1,\ldots,i_n}| = (x^1_{i_1+1} - x^1_{i_1})(x^2_{i_2+1} - x^2_{i_2}) \cdots (x^n_{i_n+1} - x^n_{i_n})$$

For $f : A \to \mathbb{R}$ a bounded function we define the lower and upper sums with respect to $\mathcal{P}$ in the obvious way:

$$L(f, \mathcal{P}) = \sum_{\pi \in \mathcal{P}} \inf_{x \in \pi} f(x) \, |\pi| \, ,$$
$$U(f, \mathcal{P}) = \sum_{\pi \in \mathcal{P}} \sup_{x \in \pi} f(x) \, |\pi| \, ,$$

where the sums are understood to be taken over all subdomains $\pi_{i_1,\ldots,i_n}$ of $\mathcal{P}$. From here, we can proceed as in Section 4.2.3, defining integrable functions and establishing the obvious generalisations of the results we found for the integral in two dimensions.

4.3 Green's Theorem in the plane

When we considered the integral in one dimension, one of the crucial results we obtained was the fundamental theorem of calculus, in particular Corollary 2.14. This states that if $f : [a, b] \to \mathbb{R}$ is continuous and $F : [a, b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all all $x \in (a, b)$ then:

$$\int_a^b f(t) dt = F(b) - F(a).$$

This result is the simplest of a family of results which relate integrals over a region $A \subset \mathbb{R}^n$ with quantities evaluated on the boundary ∂A. The general result is known as Stokes' Theorem, and is a deep and very important result in differential geometry. We shall establish a version for planar regions, known as Green's Theorem in the plane. This states that certain integrals over two-dimensional regions can be related to line integrals around the boundary of the region. We won't make any attempt to prove an optimal result.

4.3.1 Proof for simple domains

We say that a domain $A \subset \mathbb{R}^2$ is of *Type I* if there exist $a, b \in \mathbb{R}$ and continuous functions $\alpha, \beta : [a, b] \to \mathbb{R}$ such that:

$$A = \{(x, y)^t : a \leq x \leq b, \ \alpha(x) \leq y \leq \beta(x)\},$$

and moreover α and β are differentiable at all $x \in (a, b)$ with $\alpha', \beta' : (a, b) \to \mathbb{R}$ piecewise continuous.

Similarly, a domain $A \subset \mathbb{R}^2$ is of *Type II* if there exist $c, d \in \mathbb{R}$ and continuous functions $\lambda, \mu : [c, d] \to \mathbb{R}$ such that:

$$A = \{(x, y)^t : \lambda(y) \leq x \leq \mu(y), c \leq y \leq d\}.$$

and moreover λ and μ are differentiable at all $x \in (c, d)$ with $\lambda', \mu' : (c, d) \to \mathbb{R}$ piecewise continuous. See Figure 4.5 for examples of Type I and Type II domains.

A domain is *simple* if it is of both Type I and Type II, see Figure 4.6. We shall assert without proof the fact that in all three cases, A is Jordan measurable, so:

$$\int_A f(x)dx$$

makes sense whenever f is continuous on A

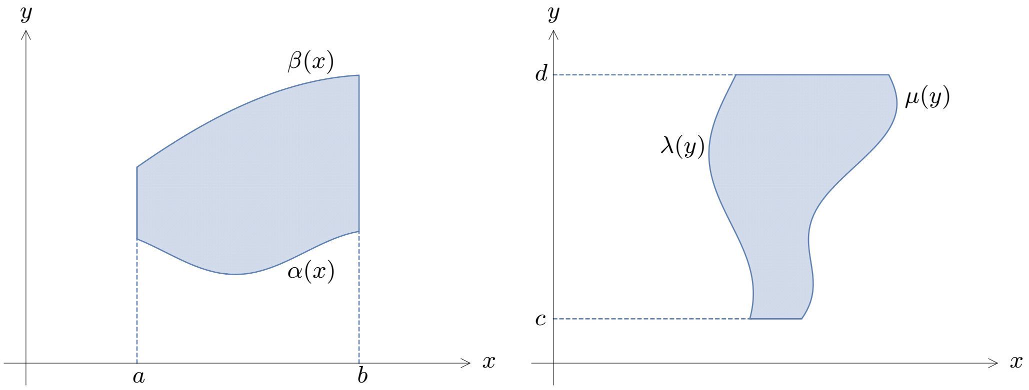

Figure 4.5 A type I (left) and type II (right) domain.

Lemma 4.15. *Suppose that $\Omega \subset \mathbb{R}^2$ is an open set, and that $A \subset \Omega$ is a type I domain. Suppose that $f : \Omega \to \mathbb{R}$ is differentiable on Ω, with continuous partial derivatives. Then:*

$$\int_A D_2 f(x)dx = \int_a^b f(t, \beta(t))dt - \int_a^b f(t, \alpha(t))dt.$$

Proof. We first apply Fubini's theorem to reduce the integral over A to a repeated integral:

$$\int_A D_2 f(x)dx = \int_a^b \left(\int_{\alpha(t)}^{\beta(t)} D_2 f(t, s)ds \right) dt. \tag{4.2}$$

Recalling that:

$$D_2 f(t, s) = \lim_{h \to 0} \frac{f(t, s + h), -f(t, s)}{h},$$

we note that the first integral may be written:

$$\int_{\alpha(t)}^{\beta(t)} D_2 f(t, s)ds = \int_{\alpha(t)}^{\beta(t)} F_t'(s)ds,$$

where $F_t : [\alpha(t), \beta(t)] \to \mathbb{R}$ is the map $s \mapsto f(t,s)$. Applying the fundamental theorem of calculus, we have:

$$\int_{\alpha(t)}^{\beta(t)} D_2 f(t,s)ds = f(t,\beta(t)) - f(t,\alpha(t)).$$

Inserting this into (4.2), we deduce

$$\int_A D_2 f(x)dx = \int_a^b \left(f(t,\beta(t)) - f(t,\alpha(t))\right) dt,$$

and the result follows. □

Lemma 4.16. *Suppose that $\Omega \subset \mathbb{R}^2$ is an open set, and that $A \subset \Omega$ is a type II domain. Suppose that $g : \Omega \to \mathbb{R}$ is differentiable on Ω, with continuous partial derivatives. Then:*

$$\int_A D_1 g(x)dx = \int_c^d g(\mu(s),s)ds - \int_c^d g(\lambda(s),s)ds.$$

Proof. The proof proceeds almost exactly as in the previous Lemma. We first apply Fubini's theorem to reduce the integral over A to a repeated integral:

$$\int_A D_1 g(x)dx = \int_c^d \left(\int_{\lambda(s)}^{\mu(s)} D_1 g(t,s)dt \right) ds. \tag{4.3}$$

Recalling that:

$$D_1 g(t,s) = \lim_{h\to 0} \frac{g(t+h,s), -g(t,s)}{h},$$

we note that the first integral may be written:

$$\int_{\lambda(s)}^{\mu(s)} D_1 g(t,s)dt = \int_{\lambda(s)}^{\mu(s)} G_s'(t)dt,$$

where $G_s : [\lambda(s), \mu(s)] \to \mathbb{R}$ is the map $t \mapsto g(t,s)$. Applying the fundamental theorem of calculus, we have:

$$\int_{\lambda(s)}^{\mu(s)} D_1 g(t,s)dt = g(\mu(s),s) - g(\lambda(s),s).$$

Inserting this into (4.3), we deduce

$$\int_A D_1 g(x)dx = \int_c^d \left(g(\mu(s),s) - g(\lambda(s),s)\right) ds,$$

and the result follows. □

Combining these two results, we arrive at Green's theorem:

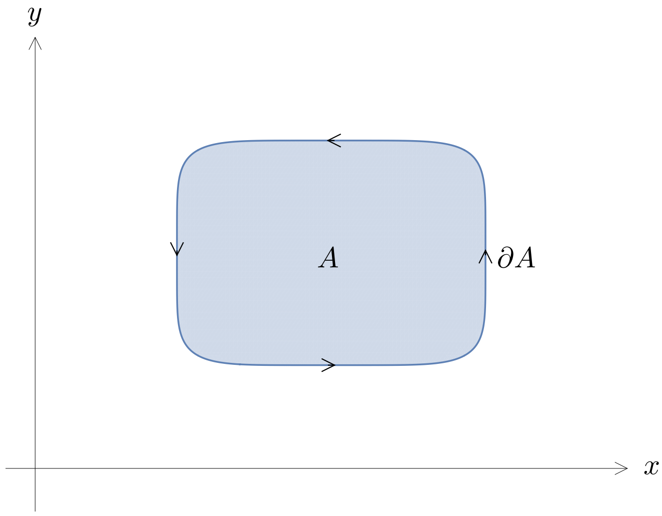

Figure 4.6 A simple domain, with the direction of the boundary indicated.

Theorem 4.17 (Green's Theorem in the plane)**.** *Suppose that $\Omega \subset \mathbb{R}^2$ is an open set, and that $A \subset \Omega$ is a simple domain. Suppose that $V : \Omega \to \mathbb{R}^2$ is differentiable at all points of Ω, with continuous partial derivatives. Then:*

$$\int_{\partial A} \langle V, d\boldsymbol{\ell} \rangle = \int_A \left[D_1 V^2(x) - D_2 V^1(x) \right] dx, \tag{4.4}$$

where we understand the integration over the boundary to be taken in the anti-clockwise sense.

Proof. 1. First, let us assume that $V^2 = 0$ on Ω. We make use of the fact that A is of Type I and apply Lemma 4.15 to the second term on the right of (4.4). We deduce:

$$\int_A -D_2 V^1(x) dx = \int_a^b V^1(t, \alpha(t)) dt - \int_a^b V^1(t, \beta(t)) dt.$$

The boundary ∂A of a Type I domain consists of at most four components (see Figure). We can parameterise these as:

$$\begin{aligned}
\gamma_1(t) &= (b, t)^t, & \alpha(b) &\leq t \leq \beta(b) \\
\gamma_2(t) &= (-t, \beta(-t))^t & -b &\leq t \leq -a \\
\gamma_3(t) &= (a, -t)^t, & -\beta(a) &\leq t \leq -\beta(a) \\
\gamma_4(t) &= (t, \alpha(t))^t, & a &\leq t \leq b.
\end{aligned}$$

so that we can compute:

$$\begin{aligned}
\gamma_1'(t) &= (0, 1)^t, & \alpha(b) &\leq t \leq \beta(b) \\
\gamma_2'(t) &= (-1, -\beta'(-t))^t & -b &\leq t \leq -a \\
\gamma_3'(t) &= (0, -1)^t, & -\beta(a) &\leq t \leq -\beta(a) \\
\gamma_4'(t) &= (1, \alpha'(t))^t, & a &\leq t \leq b.
\end{aligned}$$

Accordingly, we compute:

$$\int_{\gamma_1} \langle V, d\boldsymbol{\ell} \rangle = 0,$$
$$\int_{\gamma_2} \langle V, d\boldsymbol{\ell} \rangle = \int_{-b}^{-a} -V^1(-t, \beta(-t))dt = -\int_a^b V^1(t, \beta(t))dt,$$
$$\int_{\gamma_3} \langle V, d\boldsymbol{\ell} \rangle = 0,$$
$$\int_{\gamma_4} \langle V, d\boldsymbol{\ell} \rangle = \int_a^b V^1(t, \alpha(t))dt.$$

Putting this all together, we conclude that:

$$\int_{\partial A} \langle V, d\boldsymbol{\ell} \rangle = \int_A \left[D_1 V^2(x) - D_2 V^1(x) \right] dx,$$

when $V^2 = 0$.

2. Next, we consider the case when $V^1 = 0$ on Ω. We make use of the fact that A is of Type II and apply Lemma 4.16 to the first term on the right of (4.4). We deduce:

$$\int_A D_1 V^2(x) dx = \int_c^d \left(V^2(\mu(s), s) - V^2(\lambda(s), s) \right) ds$$

The boundary ∂A of a Type II domain consists of at most four components (see Figure). We can parameterise these as:

$$\begin{aligned}
\gamma_1(t) &= (\mu(s), s)^t, && c \le s \le d \\
\gamma_2(t) &= (-s, d)^t && -\mu(d) \le s \le -\lambda(d) \\
\gamma_3(t) &= (\lambda(-s), -s)^t, && -d \le s \le -c \\
\gamma_4(t) &= (s, c)^t, && \lambda(c) \le s \le \mu(c).
\end{aligned}$$

so that we can compute:

$$\begin{aligned}
\gamma_1'(t) &= (\mu'(s), 1)^t, && c \le s \le d \\
\gamma_2'(t) &= (-1, 0)^t && -\mu(d) \le s \le -\lambda(d) \\
\gamma_3'(t) &= (-\lambda'(-s), -1)^t, && -d \le s \le -c \\
\gamma_4'(t) &= (1, 0)^t, && \lambda(c) \le s \le \mu(c).
\end{aligned}$$

Accordingly, we compute:

$$\int_{\gamma_1} \langle V, d\boldsymbol{\ell} \rangle = \int_c^d V^2(\mu(s), s) ds,$$
$$\int_{\gamma_2} \langle V, d\boldsymbol{\ell} \rangle = 0,$$
$$\int_{\gamma_3} \langle V, d\boldsymbol{\ell} \rangle = \int_{-d}^{-c} V^2(\lambda(-s), -s) ds = -\int_c^d V^2(\lambda(s), s) ds$$
$$\int_{\gamma_4} \langle V, d\boldsymbol{\ell} \rangle = 0.$$

Putting this all together, we conclude that:

$$\int_{\partial A} \langle V, d\boldsymbol{\ell} \rangle = \int_A \left[D_1 V^2(x) - D_2 V^1(x) \right] dx,$$

when $V^1 = 0$.

3. Now, note that for any vector field V we can write $V = U + W$ for two vector fields U, W with $U^2 = W^1 = 0$ on Ω. Applying the results above together with the linearity of the integral, we deduce that:

$$\int_{\partial A} \langle V, d\boldsymbol{\ell} \rangle = \int_A \left[D_1 V^2(x) - D_2 V^1(x) \right] dx,$$

holds for any vector field V satisfying the conditions of the theorem. □

Note that while the condition of being simple is not by itself sufficient to ensure that ∂A is piecewise C^1, we nevertheless have that the integral:

$$\int_{\partial A} \langle V, d\boldsymbol{\ell} \rangle$$

always makes sense.

Example 4.5. Let's consider $A = \overline{B_1(0)}$, the unit disc centred at the origin, and we'll consider:

$$V : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \end{pmatrix}.$$

The boundary ∂A is the unit circle traversed in an anti-clockwise direction. We've already computed the integral of this vector field along the unit circle in Example 4.3 and we found:

$$\int_{\partial A} \langle V, d\boldsymbol{\ell} \rangle = 2\pi.$$

On the other hand, we can compute:

$$D_1 V^2 - D_2 V^1 = 2,$$

so that:

$$\int_A 2 dx = 2\pi.$$

Now, the disc is an example of a simple domain (check this) and so we deduce that the Jordan measure of the disc is $m(\overline{B_1(0)}) = \pi$.

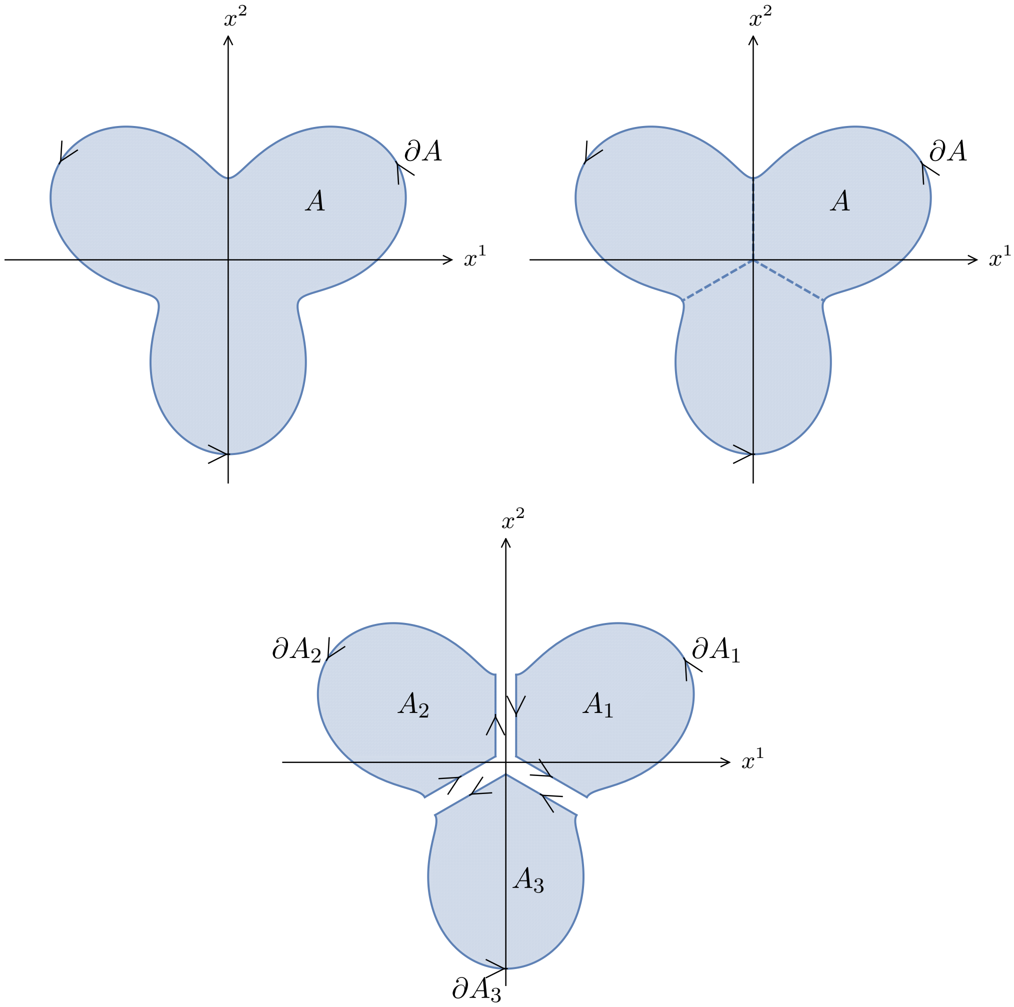

Figure 4.7 Breaking up a region into simple components.

4.3.2 Green's theorem in more general domains

The assumption that the domain is simple is rather a strong one. We can, however extend the results to more complicated domains by breaking them up into simple domains. We give here one example where this is possible, but the same basic idea can be applied in many other situations.

Consider the planar domain A shown in Figure 4.7. This is not of Type I or Type II. We can, however, split it up into three simple domains by making the cuts shown in the second diagram. The final diagram shows the three new domains A_1, A_2, A_3 slightly displaced to show the boundaries of these regions. If $f : A \to \mathbb{R}$ is any function which is integrable over A, then clearly:

$$\int_A f(x)dx = \int_{A_1} f(x)dx + \int_{A_1} f(x)dx + \int_{A_1} f(x)dx$$

Moreover, if V is any integrable vector field defined in a neighbourhood of A, then we have:

$$\int_{\partial A} \langle V, d\boldsymbol{\ell}\rangle = \int_{\partial A_1} \langle V, d\boldsymbol{\ell}\rangle + \int_{\partial A_2} \langle V, d\boldsymbol{\ell}\rangle + \int_{\partial A_3} \langle V, d\boldsymbol{\ell}\rangle$$

since the contributions from the cuts cancel, as the line segments are traversed twice in opposite directions. On each of the domains A_i, we can apply Green's theorem separately, so we deduce that Green's theorem holds for the domain A.

4.3.3 Proof of symmetry of mixed partial derivatives

When we discussed partial derivatives, we stated the following result without proving it:

Theorem 3.8 (Schwartz' Theorem)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}$ is differentiable at each $p \in \Omega$. Suppose further that for some $i, j \in \{1, \ldots, n\}$ the second partial derivatives:*

$$D_i D_j f(p), \qquad D_j D_i f(p),$$

exist and are continuous at all $p \in \Omega$. Then:

$$D_i D_j f(p) = D_j D_i f(p).$$

We are now in a position to establish this result. We first need to establish the following Lemma

Lemma 4.18. *Let $\Omega \subset \mathbb{R}^2$ be open and let $f : \Omega \to \mathbb{R}$ be continuous. Suppose that:*

$$\int_{B_r(p)} f(x)dx = 0$$

for all $r > 0, p \in \Omega$ such that the disc $B_r(p) \subset \Omega$. Then $f = 0$ on Ω.

Proof. Suppose not, there there exists $p \in \Omega$ such that $f(p) = c \neq 0$. Without loss of generality we can assume $c > 0$ (otherwise apply the argument to $-f$). Since Ω is open, there exists $r > 0$ such that $B_r(p) \subset \Omega$. Since f is continuous, there exists $\delta < r$ such that if $||x - p|| < \delta$ then $|f(x) - f(p)| < \frac{c}{2}$. Thus, we have that for $x \in B_\delta(p)$ that $f(x) \geq \frac{c}{2} > 0$. We deduce that:

$$\int_{B_\delta(p)} f(x)dx \geq \pi\delta^2 \frac{c}{2} > 0,$$

contradicting the assumption that the integral of f over any disc vanishes. □

Now we are ready to prove the result.

Proof of Theorem 3.8. First note that it's enough to show the result for $n = 2$, since for higher n we can treat all of the variables except x^i, x^j as constant. Thus, suppose that $\Omega \subset \mathbb{R}^2$ is open and $f : \Omega \to \mathbb{R}$ is differentiable at each $p \in \Omega$. Suppose further that the second partial derivatives:

$$D_1 D_2 f(x), \qquad D_2 D_1 f(x),$$

exist and are continuous at all $x \in \Omega$. Let us fix $r > 0, p \in \Omega$ such that $B_r(p) \subset \Omega$. We compute using Green's theorem:

$$\int_{B_r(p)} \left(D_1 D_2 f(x) - D_2 D_1 f(x)\right) dx = \int_{\partial B_r(p)} \langle V, d\ell \rangle ,$$

where V is the vector field:

$$V(x) = \begin{pmatrix} D_1 f(x) \\ D_2 f(x) \end{pmatrix}.$$

Now, note that if $\gamma : [a, b] \to \mathbb{R}^2$ is an anti-clockwise parameterisation of $B_r(p)$, then:

$$\begin{aligned} \left\langle V(\gamma(t)), \gamma'(t) \right\rangle &= D_1 f(\gamma(t)) \gamma^{1\prime}(t) + D_2 f(\gamma(t)) \gamma^{2\prime}(t) \\ &= (f \circ \gamma)'(t), \end{aligned}$$

by the chain rule. Applying the fundamental theorem of calculus, we have:

$$\begin{aligned} \int_{\partial B_r(p)} \langle V, d\ell \rangle = \int_a^b \left\langle V(\gamma(t)), \gamma'(t) \right\rangle dt &= \int_a^b (f \circ \gamma)'(t) dt \\ &= f(\gamma(b)) - f(\gamma(a)) = 0, \end{aligned}$$

since $\partial B_r(p)$ is closed, so $\gamma(a) = \gamma(b)$. We conclude that:

$$\int_{B_r(p)} \left(D_1 D_2 f(x) - D_2 D_1 f(x)\right) dx = 0.$$

Since $B_r(p)$ was arbitrary, by the previous Lemma we deduce that:

$$D_1 D_2 f(x) - D_2 D_1 f(x) = 0$$

for all $x \in \Omega$, which is the result we require. □

4.3.4 Change of variables formula in two dimensions

From the fundamental theorem of calculus in one dimension we were able to establish the change of variables formula (Theorem 2.15). We will show how it's possible to use Green's theorem to establish a change of variables theorem in two dimensions. We shall make strong assumptions in order to establish this result. It is certainly the case that these assumptions may be weakened. In particular the restriction we make on the shape of the domain is much stronger than necessary.

Theorem 4.19. *Let $A = [a_1, b_2] \times [a_2, b_2]$ be a square domain in $\mathbb{R}^2$ and let $f : A \to \mathbb{R}$ be continuous. Suppose $\Omega \subset \mathbb{R}^2$ be open, and suppose $\phi : \Omega \to A$ is an injective, orientation preserving*[2]*, twice continuously differentiable function. Finally suppose that $D \subset \Omega$ has the property that Green's theorem is valid on both D and $\phi(D)$. Then:*

$$\int_{\phi(D)} f(x) dx = \int_D f \circ \phi(x) \left| D\phi(x) \right| dx$$

Here $|D\phi|$ indicates the determinant of the Jacobian.

[2] A map is orientation preserving if $|D\phi| > 0$ everywhere

(*) Proof. We shall proceed by applying Green's theorem to relate the left hand side to a line integral around $\partial\phi(D)$ and then showing that this can be transformed into a line integral over ∂D. Finally another application of Green's theorem gives the result.

To set up some notation let us assume that $\gamma : [a, b] \to \mathbb{R}^2$ is the boundary of D, traversed in an anti-clockwise direction. The boundary of D will be $\tilde{\gamma} = \phi \circ \gamma : [a, b] \to \mathbb{R}^2$. The condition that ϕ is orientation preserving is required to ensure that $\tilde{\gamma}$ traverses ∂D in the correct sense.

1. First, we note that since f is continuous, there exists a continuous function $F : A \to \mathbb{R}$ such that $D_1F(x) = f(x)$ for all $x \in A$, which we can define by:

$$F(t, s) = \int_{a_1}^{t} f(t', s)dt'.$$

Now, applying Green's theorem on the domain ∂D to the vector field:

$$V(x) = \begin{pmatrix} 0 \\ F(x) \end{pmatrix},$$

we have:

$$\begin{aligned}\int_{\phi(D)} f(x)dx &= \int_{\phi(D)} D_1F(x)dx \\ &= \int_{\partial\phi(D)} \langle V, d\ell\rangle \\ &= \int_a^b F \circ \tilde{\gamma}(t)\, \tilde{\gamma}^{2\prime}(t)dt.\end{aligned}$$

2. Next, we can apply the chain rule to relate $\tilde{\gamma}^{2\prime}(t)$ to γ by:

$$\tilde{\gamma}^{2\prime}(t) = D_1\phi^2(\gamma(t))\gamma^{1\prime}(t) + D_2\phi^2(\gamma(t))\gamma^{2\prime}(t),$$

so that:

$$\begin{aligned}\int_a^b F \circ \tilde{\gamma}(t)\, \tilde{\gamma}^{2\prime}(t)dt &= \int_a^b F \circ \phi(\gamma(t)) \left[D_1\phi^2(\gamma(t))\gamma^{1\prime}(t) + D_2\phi^2(\gamma(t))\gamma^{2\prime}(t)\right] dt \\ &= \int_a^b \left\langle \begin{pmatrix} F \circ \phi\ D_1\phi^2 \\ F \circ \phi\ D_2\phi^2 \end{pmatrix} (\gamma(t)), \gamma'(t) \right\rangle dt \\ &= \int_{\partial D} \langle W, d\ell\rangle,\end{aligned}$$

where W is the vector field:

$$W(x) = \begin{pmatrix} F \circ \phi(x)\ D_1\phi^2(x) \\ F \circ \phi(x)\ D_2\phi^2(x) \end{pmatrix}.$$

3. Now we compute with the chain rule:

$$D_1W^2(x) = \big(D_1F(\phi(x))D_1\phi^1(x) + D_2F(\phi(x))D_1\phi^2(x)\big)\, D_2\phi^2(x) + F(\phi(x))D_1D_2\phi(x),$$
$$D_2W^1(x) = \big(D_1F(\phi(x))D_2\phi^1(x) + D_2F(\phi(x))D_2\phi^2(x)\big)\, D_1\phi^2(x) + F(\phi(x))D_2D_1\phi(x),$$

so that:

$$\begin{aligned}\int_{\partial D} \langle W, d\ell\rangle &= \int_D \big(D_1W^2(x) - D_2W^1(x)\big)\, dx \\ &= \int_D D_1F(\phi(x))\left[D_1\phi^1(x)D_2\phi^2(x) - D_2\phi^1(x)D_1\phi^2(x)\right] dx \\ &= \int_D f\circ\phi(x)\ |D\phi(x)|\, dx,\end{aligned}$$

where we use the fact that $D_1F = f$. Combining all of these identities, the result follows. □

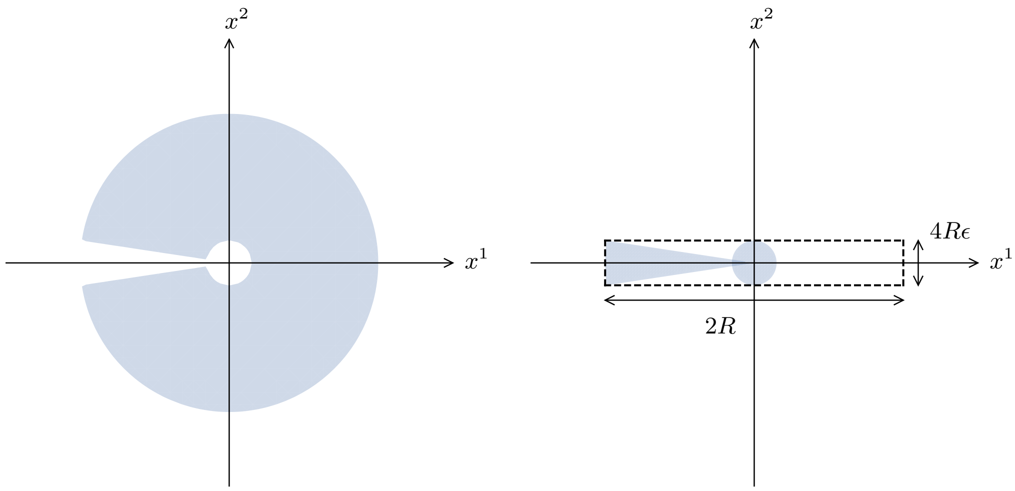

Figure 4.8 The regions $\phi(D)$, and $B_R(0)\setminus\phi(D)$.

Example 4.6. Fix $R > 0$. Let $A = [-2R, 2R] \times [-2R, 2R]$ and suppose that $f : A \to \mathbb{R}$ is continuous. Fix $\epsilon > 0$, and let us take $\Omega = (\epsilon R, 2R) \times (-\pi + \epsilon, \pi - \epsilon)$ The map:

$$\phi : \begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

is injective and smooth, and we can compute for $z = (r, \theta)^t$:

$$\begin{aligned}|D\phi(z)| &= D_1\phi^1(z)D_2\phi^2(z) - D_1\phi^2(z)D_2\phi^1(z) \\ &= \cos\theta \times r\cos\theta - \sin\theta \times (-r\cos\theta) \\ &= r.\end{aligned}$$

Let us take $D = [2\epsilon R, R] \times [-\pi + 2\epsilon, \pi - 2\epsilon]$. We can apply the above result to conclude (making use of Fubini's theorem):

$$\int_{\phi(D)} f(x)dx = \int_D f \circ \phi(z) \ |D\phi(z)| \, dz = \int_{2\epsilon R}^{R} \left(\int_{-\pi+2\epsilon}^{\pi-2\epsilon} f(r\cos\theta, r\sin\theta)d\theta \right) rdr.$$

Here we are justified in using Fubini since the function $(r, \theta) \mapsto f(r\cos\theta, r\sin\theta)$ is continuous (as it is a composition of continuous functions).

Now, let us consider the domain $B_R(0) \setminus \phi(D)$. We can estimate the area of this set by putting it inside a box of height $4R\epsilon$ and width $2R$, so that $|B_R(0) \setminus \phi(D)| \leq 8\epsilon R^2$. Since f is a continuous function on a bounded set, we have $|f(x)| \leq K$ for some K. We deduce:

$$\left| \int_{B_R(0)} f(x)dx - \int_{\phi(D)} f(x)dx \right| \leq 8\epsilon K R^2$$

Since ϵ was arbitrary, we deduce that:

$$\int_{B_R(0)} f(x)dx = \int_0^R \left(\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta)d\theta \right) rdr.$$

which is the usual change of variables formula for polar coordinates in the plane. Notice that we had to introduce ϵ in order that the conditions of Theorem 4.19 are satisfied. This type of approach enables us to extend the change of variables formula in other cases where the hypotheses of Theorem 4.19 are not satisfied.

Exercise 9.6. a) Show that the improper integral:

$$\int_1^\infty e^{-u} du$$

converges.

b) By making use of the estimate:

$$\int_1^R e^{-u^2} du \leq \int_1^R e^{-u} du, \qquad \text{for } R > 1,$$

or otherwise, establish that the improper integral:

$$I := \int_{-\infty}^{\infty} e^{-u^2} du$$

converges.

c) Let $S_R = [-R, R] \times [-R, R]$. Show that for any function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f \geq 0$, the estimate:

$$\int_{B_R(0)} f(x)dx \leq \int_{S_R} f(x)dx \leq \int_{B_{\sqrt{2}R}(0)} f(x)dx$$

holds.

d) By taking $f : (u, v)^t \mapsto e^{-u^2-v^2}$, or otherwise, show that:

$$\pi\left(1 - e^{-R^2}\right) \leq \int_{-R}^{R}\left(\int_{-R}^{R} e^{-u^2-v^2}\,du\right) dv \leq \pi\left(1 - e^{-2R^2}\right)$$

e) Deduce that:

$$I = \sqrt{\pi}.$$