Chapter 1

Uniform continuity and convergence

So far, we have considered sequences of points in $\mathbb{R}^n$ and their limits. For many applications, we want to consider sequences of other objects, and to talk about their limits. For example it is often useful to consider sequences of *functions*. We might imagine that if we have a suitable limit of continuous (or differentiable) functions, then the limit should be a continuous (resp. differentiable) function. In certain circumstances this is true, and in this chapter we will develop some of the machinery required to establish such a result.

1.1 Uniform continuity

1.1.1 Definition and examples

In the previous section (and in last year's courses) we defined what it means for a function to be continuous at a point p. We said that the function is continuous on some set $A \subset \mathbb{R}^n$ if it is continuous at all $p \in A$. This is a very useful concept, however in certain circumstances we need a slightly stronger property, called *uniform continuity*.

Definition 1.1. Suppose $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. We say that f is *uniformly continuous* on A if the following holds: given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$ with $||x - y|| < \delta$ we have:

$$||f(x) - f(y)|| < \epsilon.$$

For comparison, let's recall the definition of continuity at a point:

Definition I.6. Let $A \subset \mathbb{R}^n$ and suppose $f : A \to \mathbb{R}^m$. We say that f is continuous at $p \in A$ if the following holds: given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ with $||x - p|| < \delta$ we have:

$$||f(x) - f(p)|| < \epsilon.$$

We see that the crucial difference between the definition for a function continuous on A and a function *uniformly* continuous on A is that in the first case δ is allowed to depend on the point p at which we're testing continuity. In the second case, one δ has to work at every point in A.

Lemma 1.1. *Let $A \subset \mathbb{R}^n$ and suppose $f : A \to \mathbb{R}^m$. If f is uniformly continuous on A, then f is continuous on A.*

Proof. Fix $p \in A$ and $\epsilon > 0$. By the uniform continuity of f, there exists $\delta > 0$ such that if $||x - p|| < \delta$, then:

$$||f(x) - f(p)|| < \epsilon.$$ □

Example 1.1. Suppose $\alpha \in \mathbb{R}$ and $z \in \mathbb{R}^n$. The function:

$$\begin{array}{cccc} f & : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & & x & \mapsto & \alpha x + z \end{array}$$

is uniformly continuous. To see this, suppose that $||x - y|| < \delta$. Then we have:

$$||f(x) - f(y)|| = ||\alpha(x - y)|| = |\alpha| \, ||x - y|| \leq |\alpha| \, \delta.$$

Taking $\delta = \frac{\epsilon}{1+|\alpha|}$, we have the result. Note that our choice of δ does not depend on x, y.

Example 1.2. The function:

$$\begin{array}{cccc} f & : & (0, \infty) & \to & \mathbb{R} \\ & & x & \mapsto & \frac{1}{x} \end{array}$$

is continuous on $(0, \infty)$, but *not* uniformly continuous. The continuity of f at each $p > 0$ follows from Corollary I.8 *iii*). To see that f is not uniformly continuous, it is enough to show that for any $\delta > 0$ there exist $x, y \in (0, \infty)$ with $||x - y|| < \delta$ and $||f(x) - f(y)|| > 1$. Consider for some $0 < \kappa < 1$:

$$x = \frac{\kappa}{2}\delta, \quad y = \kappa\delta, \quad \implies \quad ||x - y|| = \frac{\kappa}{2}\delta < \delta.$$

On the other hand,

$$||f(x) - f(y)|| = \frac{1}{\kappa\delta},$$

so if $\kappa = \frac{1}{1+\delta}$ we have $||f(x) - f(y)|| > 1$.

The key idea of uniform continuity is that points which are close together in the domain end up close together after they've been mapped to the range by f, and this should happen *independently of where in the range the points are.* In the second example above, we see that two points which are close together and near 0 are mapped much further apart than two points close together and near 1.

Exercise 2.6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable (hence continuous) everywhere on $\mathbb{R}$ and the derivative satisfies:

$$\left|f'(x)\right| \leq M, \qquad \text{for all } x \in \mathbb{R}.$$

a) Show that for $x, y \in \mathbb{R}$ we have the estimate:

$$|f(x) - f(y)| \leq M \, |x - y| \,. \tag{1.1}$$

b) Deduce that f is uniformly continuous on $\mathbb{R}$.

The condition (1.1), which can be imposed whether or not f is differentiable, is known as the Lipschitz condition.

Exercise 2.7. Which of the following functions is *i*) continuous, *ii*) uniformly continuous on its domain?

a) $\begin{array}{cccc} f & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & e^x \end{array}$

b) $\begin{array}{cccc} f & : & (0,1) & \to & \mathbb{R} \\ & & x & \mapsto & e^x \end{array}$

c) $\begin{array}{cccc} f & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & \sin x \end{array}$

d) $\begin{array}{cccc} f & : & [0,\infty) & \to & \mathbb{R} \\ & & x & \mapsto & \sqrt{x} \end{array}$

*f) $\begin{array}{cccc} f & : & \mathbb{R}^n \setminus \{0\} & \to & \mathbb{R}^n \\ & & x & \mapsto & \frac{x}{||x||} \end{array}$

Exercise 2.8. Let $A \subset \mathbb{R}^n$ and suppose that $f, g : A \to \mathbb{R}$.

a) Show that if f, g are uniformly continuous and bounded, then so is fg.

b) Show that if f, g are uniformly continuous but not necessarily bounded, then fg need not be uniformly continuous.

1.1.2 Continuous functions on a compact set

The following result is very useful, as it allows us to deduce that a continuous function is uniformly continuous, just based on knowledge about the domain of the function:

Theorem 1.2. *Suppose that $E \subset \mathbb{R}^n$ is compact and $f : E \to \mathbb{R}^m$ is continuous on E. Then f is* uniformly *continuous on E.*

Proof. Suppose f is not uniformly continuous. Then[1] there exists an $\epsilon > 0$ such that for any $i \in \mathbb{N}$ we can find x_i, y_i with:

$$||x_i - y_i|| < \frac{1}{2^i}, \qquad ||f(x_i) - f(y_i)|| \geq \epsilon.$$

Now $(x_i)_{i=0}^\infty$ is a sequence in a compact set, so by Bolzano–Weierstrass it has a convergent subsequence $(x_{i_j})_{j=0}^\infty$ with $x_{i_j} \to x \in E$. Next look at $(y_{i_j})_{j=0}^\infty$. Again this is a sequence in a compact set, so it has a convergent subsequence $(y_{i_{j_l}})_{l=0}^\infty$ with $y_{i_{j_l}} \to y$. Relabelling

[1] Check you understand why this is!

these sequences $u_l = x_{i_{j_l}}$, $w_l = y_{i_{j_l}}$, we have $(u_l)_{l=0}^{\infty}$ with $u_l \to x$ and $(w_l)_{l=0}^{\infty}$ with $w_l \to y$. Furthermore, since $i_{j_l} \geq j_l \geq l$, we have:

$$||u_l - w_l|| < \frac{1}{2^l}, \qquad ||f(u_l) - f(w_l)|| \geq \epsilon.$$

Sending $l \to \infty$, we conclude from the first inequality that $||x - y|| = 0$, so $x = y$. From the second inequality, using the continuity of f, we have $||f(x) - f(y)|| \geq \epsilon$, which is a contradiction. □

1.2 Uniform convergence

Now we want to think a bit about convergence of sequences of functions. When we say that a sequence $(x_i)_{i=0}^{\infty}$ with $x_i \in \mathbb{R}^n$ converges to $x \in \mathbb{R}^n$, crudely we're saying:

> "By going far enough along the sequence, we can ensure we're as close as we like to x".

For points in $\mathbb{R}^n$, it's pretty unambiguous what we mean by "close enough": we want the points in the sequence to eventually be a small distance away from x. For functions, it turns out that there are different plausible notions of "close enough". Which one we use depends to an extent on the problem that we wish to address.

1.2.1 Definition and examples

We will start with the most naïve definition possible for the convergence of a sequence of functions.

Definition 1.2. Let $A \subset \mathbb{R}^n$. Suppose $(f_i)_{i=0}^{\infty}$ is a sequence of functions $f_i : A \to \mathbb{R}^m$. We say that $f_i \to f$ *pointwise* if the following holds: given $\epsilon > 0$, for each $x \in A$ there exists $N \in \mathbb{N}$ such that for all $i \geq N$ we have:

$$||f_i(x) - f(x)|| \leq \epsilon.$$

Equivalently, we can write this as:

$$f_i(x) \to f(x) \qquad \text{for all } x \in A.$$

This definition has the advantage of being the simplest thing to write down, and it is usually relatively simple to check whether the condition holds.

Example 1.3. Consider the sequence of functions $(f_i)_{i=0}^{\infty}$ given by:

$$\begin{array}{cccc} f_i & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & x + 2^{-i}x^2. \end{array}$$

This converges pointwise to $f(x) = x$, since for any $x \in \mathbb{R}$ we have:

$$x + 2^{-i}x^2 \to x.$$

Example 1.4. For $i \in \mathbb{N}$, let $f_i : [-1,1] \to \mathbb{R}$ be given by:

$$f_i(x) = \begin{cases} -1 & x \leq -2^{-i}, \\ 2^i x & -2^{-i} < x < 2^{-i} \\ 1 & x \geq 2^{-i}, \end{cases}$$

Then $f_i \to f$ pointwise, where $f : [-1,1] \to \mathbb{R}$ is given by:

$$f(x) = \begin{cases} -1 & x < 0, \\ 0 & x = 0 \\ 1 & x > 0, \end{cases}$$

To see this, first suppose $x < 0$. Then for all sufficiently large i we have $x \leq -2^{-i}$, so $f_i(x) = -1$ and $f_i(x) \to f(x)$. Similarly, suppose $x > 0$. Then for all sufficiently large i we have $x \geq 2^{-i}$, so $f_i(x) = 1$ and $f_i(x) \to f(x)$. Finally, if $x = 0$ then $f_i(x) = 0$ for all i and so $f_i(x) \to f(x)$.

Example 1.4 shows that pointwise convergence has some drawbacks. The functions f_i are all continuous (indeed uniformly continuous), however the limiting function f is discontinuous. We often want to argue that the limit of a sequence of continuous functions is continuous and it's clear that in order to do this, we need something stronger than pointwise continuity. This is the motivation for the idea of *uniform continuity*.

Definition 1.3. Let $A \subset \mathbb{R}^n$. Suppose $(f_i)_{i=0}^{\infty}$ is a sequence of functions $f_i : A \to \mathbb{R}^m$. We say that $f_i \to f$ *uniformly* if the following holds: given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i \geq N$ we have:

$$||f_i(x) - f(x)|| \leq \epsilon, \qquad \text{for all } x \in A.$$

Notice that there is very little difference in the way that uniform convergence and pointwise convergence are defined! The key difference is in where in the statement we have "for all $x \in A$". Ultimately, the issue is that in the case of pointwise convergence, we are allowed to choose N depending on x. In the case of uniform convergence we are only allowed to choose one N and it has to work for all $x \in A$.

Notice that uniform convergence implies pointwise convergence.

Example 1.5. The sequence:

$$\begin{array}{rcccl} f_i & : & [-1,1] & \to & \mathbb{R} \\ & & x & \mapsto & x + 2^{-i}x^2. \end{array}$$

converges uniformly to $f(x) = x$. To see this, note that for any $x \in [-1,1]$

$$|f_i(x) - f(x)| = \left|2^{-i}x\right| \leq 2^{-i}.$$

For any $\epsilon > 0$ we can find N such that $2^{-i} < \epsilon$ for all $i \geq N$, so we indeed have that the convergence is uniform.

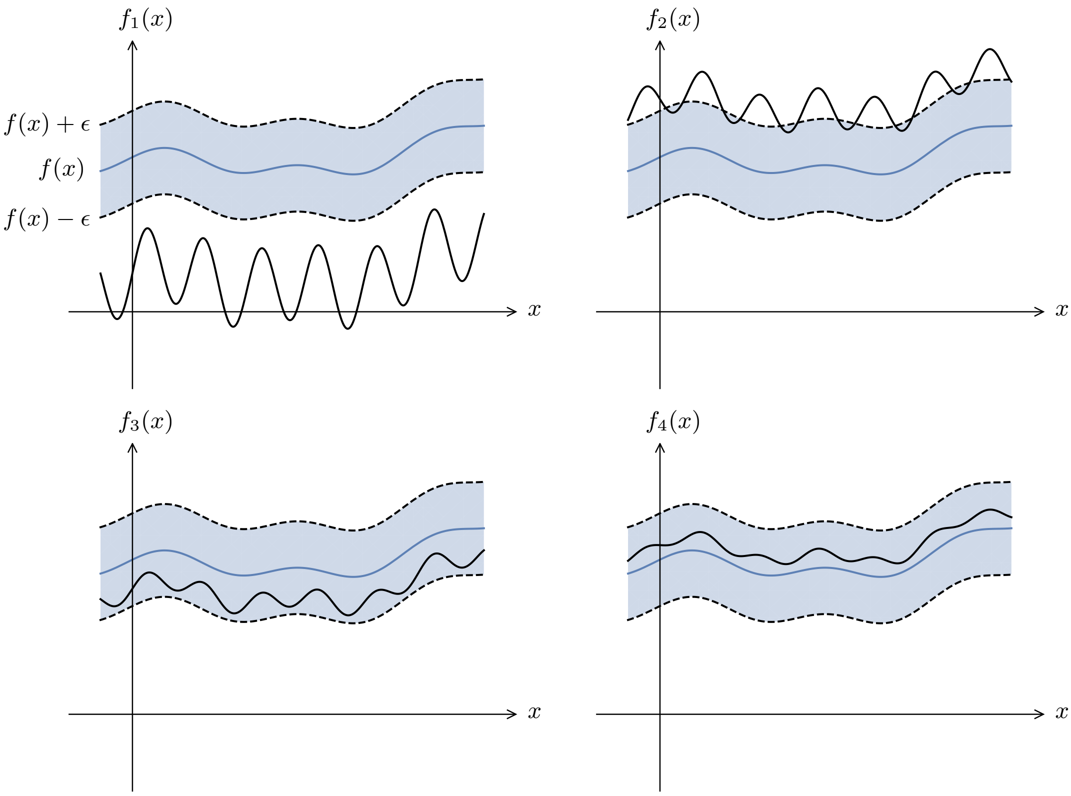

Figure 1.1 A sequence of functions converging uniformly

Example 1.6. The sequence in Example 1.4 does *not* converge uniformly. To see this, note that

$$f_i\left(2^{-(i+1)}\right) - f\left(2^{-(i+1)}\right) = \frac{1}{2} - 1 = -\frac{1}{2},$$

so that for $0 < \epsilon < \frac{1}{2}$ there can exist no $N \in \mathbb{N}$ such that for all $i \geq N$ and $x \in [-1,1]$we have:

$$||f_i(x) - f(x)|| < \epsilon.$$

The key feature of uniform convergence is the idea that $f_i(x) \to f(x)$ at (roughly) the same rate everywhere on A. We show this for a few terms of a sequence of functions converging uniformly in Figure 1.1

Exercise 3.1. For $i \in \mathbb{N}$, let $f_i : [0,1] \to \mathbb{R}$ be given by:

$$f_i(x) = \begin{cases} 2^i x & 0 \leq x < 2^{-i}, \\ 2 - 2^i x & 2^{-i} \leq x < 2^{-(i-1)} \\ 0 & x \geq 2^{-(i-1)}, \end{cases}$$

a) Sketch f_i.

b) Show that $f_i \to 0$ pointwise.

c) Is it true that:

$$\lim_{i\to\infty} \sup_{x\in[0,1]} f_i(x) = \sup_{x\in[0,1]} \lim_{i\to\infty} f_i(x)?$$

Exercise 3.2. State whether the sequence $(f_i)_{i=0}^{\infty}$ of functions converges pointwise or uniformly, when:

a) $$\begin{array}{rcrcl} f_i & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & e^x + \frac{1}{i+1} \end{array}$$

b) $$\begin{array}{rcrcl} f_i & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & x + 2^{-i}x^2 \end{array}$$

c) $f_i : [0,1] \to \mathbb{R}$, where:

$$f_i(x) = \begin{cases} 2^i x & 0 \le x < 2^{-i}, \\ 2 - 2^i x & 2^{-i} \le x < 2^{-(i-1)} \\ 0 & x \ge 2^{-(i-1)}, \end{cases}$$

Exercise 3.3. Suppose $A \subset \mathbb{R}^n$ and let $(f_i)_{i=0}^{\infty}$, where $f_i : A \to \mathbb{R}^m$ be a sequence of functions. Classify whether the following statements are equivalent to

i) $f_i \to f$ pointwise,

ii) $f_i \to f$ uniformly,

iii) neither.

a) Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i \ge N$:

$$\sup_{x\in A} ||f_i(x) - f(x)|| < \epsilon.$$

b) For all $x \in A$, for all $i \in \mathbb{N}$ there exists $\epsilon > 0$ such that:

$$||f_i(x) - f(x)|| < \epsilon.$$

c) There exists $N \in \mathbb{N}$ such that for all $x \in A$, $i \ge N$, $\epsilon > 0$:

$$||f_i(x) - f(x)|| < \epsilon.$$

*d) $\forall \epsilon > 0 \, \forall x \in A \, \exists N \in \mathbb{N} \, (i \ge N \implies ||f_i(x) - f(x)|| < \epsilon)$

*e) $\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall x \in A \, (i \ge N \implies ||f_i(x) - f(x)|| < \epsilon)$

*f) $\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall x \in A \, (i \ge N \iff ||f_i(x) - f(x)|| < \epsilon)$

Exercise 3.4. Suppose $A \subset \mathbb{R}^n$ and let $(f_i)_{i=0}^{\infty}$, where $f_i : A \to \mathbb{R}^m$ be a sequence of functions. Suppose that $f_i \to f$ uniformly on A. Show that if $B \subset A$, then $f_i \to f$ uniformly on B.

1.2.2 Uniform limits of uniformly continuous functions

One of the main motivations for introducing the idea of uniform continuity and uniform convergence is the result we shall prove in this section. It tells us that if we have a sequence of functions which are uniformly continuous and they converge uniformly, then the function they converge to must also be uniformly continuous. We shall see that this is a very useful result in many situations.

Theorem 1.3. *Let $A \subset \mathbb{R}^n$ and suppose $(f_i)_{i=0}^{\infty}$ is a sequence of functions $f_i : A \to \mathbb{R}^m$ such that:*

i) $f_i \to f$ uniformly on A.

ii) Each f_i is uniformly continuous on A,

Then $f : A \to \mathbb{R}^m$ is uniformly continuous on A.

Proof. Fix $\epsilon > 0$. Firstly, by property *i*) above, we know that there exists $N \in \mathbb{N}$ such that for all $i \geq N$ and all $x \in A$ we have:

$$||f_i(x) - f(x)|| < \frac{\epsilon}{3}.$$

Fix $k \geq N$. By property *ii*) above, we know that there exists δ such that for all $x, y \in A$ with $||x - y|| < \delta$, we have:

$$||f_k(x) - f_k(y)|| < \frac{\epsilon}{3}.$$

Now, suppose $x, y \in A$ with $||x - y|| < \delta$. We estimate:

$$\begin{aligned} ||f(x) - f(y)|| &= ||f(x) - f_k(x) + f_k(x) - f_k(y) + f_k(y) - f(y)|| \\ &\leq ||f(x) - f_k(x)|| + ||f_k(x) - f_k(y)|| + ||f_k(y) - f(y)|| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This is precisely the statement that f is uniformly continuous on A. □

Note that this theorem gives us an alternative proof that the sequence in Example 1.4 does not converge uniformly. The functions f_i are all continuous functions on a compact set (hence uniformly continuous by Theorem 1.2). If $f_i \to f$ uniformly, then f is continuous, however we know that the limit f is not continuous and thus the convergence cannot be uniform.

We can in fact relax the requirement of uniform continuity on the sequence f_i.

Corollary 1.4. *Let $U \subset \mathbb{R}^n$ be open and suppose $(f_i)_{i=0}^{\infty}$ is a sequence of functions $f_i : U \to \mathbb{R}^m$ such that:*

i) $f_i \to f$ uniformly on U.

ii) Each f_i is continuous on U,

Then $f : U \to \mathbb{R}^m$ is continuous on U.

Proof. Fix $p \in U$. Since U is open, there exists r such that $B_{2r}(p) \subset U$, so in particular $\overline{B_r(p)} \subset U$. By Theorem 1.2, each f_i restricted to $\overline{B_r(p)}$ is uniformly continuous, and moreover we have that $f_i \to f$ uniformly on $\overline{B_r(p)}$. By Theorem 1.3, we have that f is uniformly continuous on $\overline{B_r(p)}$, hence continuous at p. Since p was arbitrary, we're done. □

1.2.3 (*) The space $C^0([a, b])$

Let us restrict our attention for a moment to the case of functions defined on a closed interval in $\mathbb{R}$. Let us introduce some new notation. We will say that $C^0([a, b])$ is the set of all continuous functions mapping $[a, b] \to \mathbb{R}$. In other words:

$$f \in C^0([a,b]) \iff f : [a,b] \to \mathbb{R}, \text{ and } f \text{ is continuous on } [a,b].$$

Note that by Theorem 1.2, we know that $f \in C^0([a, b])$ implies that f is uniformly continuous. We also know by Theorem I.9 that f achieves its maximum and minimum on $[a, b]$.

We can define a function on $C^0([a, b])$ as follows:

$$\begin{array}{rccl} ||\cdot||_{C^0} & : & C^0 & \to & [0,\infty) \\ & & f & \mapsto & ||f||_{C^0} := \sup_{x\in[a,b]} |f(x)|\,. \end{array}$$

In other words, $||\cdot||_{C^0}$ assigns to each element f of $C^0([a, b])$ the largest value achieved by $|f(x)|$ on the interval. We call $||\cdot||_{C^0}$ the norm on $C^0([a, b])$.

Lemma 1.5. *The the norm on $C^0([a, b])$ has the following properties:*

i) For all $f \in C^0([a, b])$, we have $||f||_{C^0} \geq 0$, with $||f||_{C^0} = 0$ if and only if $f \equiv 0$.

ii) For all $f \in C^0([a, b])$, if $a \in \mathbb{R}$, then $||af||_{C^0} = |a|\,||f||_{C^0}$

iii) The norm satisfies the triangle inequality:

$$||f+g||_{C^0} \leq ||f||_{C^0} + ||g||_{C^0}\,, \qquad \forall\ f, g, \in C^0. \tag{1.2}$$

Proof. i) It is clear that since $||f||_{C^0}$ is the supremum of a set of non-negative numbers, it must be non-negative. If $||f||_{C^0} = 0$, then we have for all $x \in [a, b]$:

$$0 \leq |f(x)| \leq \sup_{y\in[a,b]} |f(y)| = 0$$

so that $f(x) = 0$ for all $x \in [a, b]$.

ii) If $a \in \mathbb{R}$, then:

$$||af||_{C^0} = \sup_{x\in[a,b]} |af(x)| = |a| \sup_{x\in[a,b]} |f(x)| = |a|\,||f||_{C^0}\,.$$

iii) We estimate:

$$\begin{aligned}||f+g||_{C^0} &= \sup_{x\in[a,b]} |f(x)+g(x)| \leq \sup_{x\in[a,b]} (|f(x)|+|g(x)|) \\ &\leq \sup_{x\in[a,b]} |f(x)| + \sup_{y\in[a,b]} |g(y)| = ||f||_{C^0} + ||g||_{C^0}\,.\end{aligned}$$

Here we have used the triangle inequality for $|\cdot|$, as well as the fact that $\sup(a+b) \leq \sup a + \sup b$.

□

With the norm on $C^0([a,b])$, we can give the definition of uniform convergence very succinctly. A sequence $(f_i)_{i=0}^{\infty}$ converges uniformly to f if: given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i \geq N$ we have:

$$||f_i - f||_{C^0} < \epsilon.$$

If we compare this with the definition of convergence in $\mathbb{R}^n$, Definition I.1, we see there are some very close parallels. This is because $\mathbb{R}^n$ with its standard norm $||\cdot||$ and $C^0([a,b])$ with the norm $||\cdot||_{C^0}$ are both examples of a more general structure, a *normed space*.

Recall from Exercise 2.1 that a sequence $(x_i)_{i=0}^{\infty}$ with $x_i \in \mathbb{R}^n$ converges if and only if it is a Cauchy sequence. A Cauchy sequence satisfies the following condition: given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i, j \geq N$ we have:

$$||x_i - x_j|| < \epsilon.$$

In other words a Cauchy sequence is one for which all of the terms are eventually close together. This fact, that Cauchy sequences converge, is a fundamental property of $\mathbb{R}^n$ called *completeness*. In fact, one can replace the least upper bound axiom of $\mathbb{R}$ with the assumption that Cauchy sequences converge: the two statements are equivalent.

We make the obvious definition of a Cauchy sequence in $C^0([a,b])$.

Definition 1.4. A sequence $(f_i)_{i=0}^{\infty}$ with $f_i \in C^0([a,b])$ is *Cauchy* if the following holds: given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i, j \geq N$ we have:

$$||f_i - f_j||_{C^0} < \epsilon.$$

Remarkably, $C^0([a,b])$ also has the completeness property enjoyed by $\mathbb{R}^n$:

Theorem 1.6. *A sequence $(f_i)_{i=0}^{\infty}$ with $f_i \in C^0([a,b])$ converges uniformly to some $f \in C^0([a,b])$ if and only if it is Cauchy.*

Proof. First, suppose $(f_i)_{i=0}^{\infty}$ converges uniformly to f. Let $\epsilon > 0$. By the convergence of f_i, we know that there exists $N \in \mathbb{N}$ such that for all $i \geq N$ we have:

$$||f_i - f||_{C^0} < \frac{\epsilon}{2}.$$

Now suppose $i, j \geq N$. Then:

$$||f_i - f_j||_{C^0} = ||f_i - f + f - f_j||_{C^0} \leq ||f_i - f||_{C^0} + ||f - f_j||_{C^0} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Next, suppose that $(f_i)_{i=0}^{\infty}$ is Cauchy. This means that given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $y \in [a,b]$ and $i, j \geq N$:

$$|f_i(y) - f_j(y)| \leq \sup_{x \in [a,b]} |f_i(x) - f_j(x)| = ||f_i - f_j||_{C^0} < \epsilon. \tag{1.3}$$

Thus, for any $y \in [a,b]$, the sequence $(f_i(y))_{i=0}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, so by the results of Exercise 2.1, $f_i(y)$ converges to some $f(y) \in \mathbb{R}$. Thus we have established that f_i converges to some f pointwise.

To complete the proof, it is enough to show that $f_i \to f$ uniformly. Repeating the argument that led to (1.3), we know that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $y \in [a,b]$ and $i, j \geq N$:

$$|f_i(y) - f_j(y)| < \frac{\epsilon}{2}.$$

Taking the limit $j \to \infty$, we conclude that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $i \geq N$ we have:

$$|f_i(y) - f(y)| \leq \frac{\epsilon}{2} < \epsilon,$$

for any $y \in [a,b]$. This is precisely the statement of uniform convergence. Since we know that a uniformly convergent sequence of uniformly continuous functions is uniformly continuous, we have that $f \in C^0([a,b])$. □

A normed space with the completeness property is called a *Banach space*. These are very important structures, and you will learn much more about them if you take the course M3P7 Functional Analysis.

Exercise 3.5 (*)**.** Let $E \subset \mathbb{R}^n$ be compact. We define $C^0(E, \mathbb{R}^m)$ to be the set of all continuous functions $f : E \to \mathbb{R}^m$, and equip $C^0(E, \mathbb{R}^m)$ with the norm:

$$\begin{array}{rccl} ||\cdot||_{C^0} & : & C^0 & \to & [0,\infty) \\ & & f & \mapsto & ||f||_{C^0} := \sup_{x \in E} ||f(x)|| . \end{array}$$

Show that $C^0(E, \mathbb{R}^m)$ is complete.

Exercise 3.6 (*)**.** Suppose that $f : (a,b) \to \mathbb{R}$ is uniformly continuous, and suppose $(x_j)_{j=0}^{\infty}$ and $(y_j)_{j=0}^{\infty}$ are two sequences of points $x_j, y_j \in (a,b)$ with $x_j \to a$ and $y_j \to b$.

a) Show that the sequences $(f(x_j))_{j=0}^{\infty}$ and $(f(y_j))_{j=0}^{\infty}$ are Cauchy.

b) Setting $f_L := \lim_{j\to\infty} f(x_j)$, $f_R := \lim_{j\to\infty} f(y_j)$, show that the function $\tilde{f} : [a,b] \to \mathbb{R}$ given by:

$$\tilde{f}(x) = \begin{cases} f_L & x = a, \\ f(x) & a < x < b \\ f_R & x = b, \end{cases}$$

is continuous on $[a,b]$.

c) Deduce that $f : (a,b) \to \mathbb{R}$ is uniformly continuous if and only if there exists a continuous function $\tilde{f} : [a,b] \to \mathbb{R}$ such that $f(x) = \tilde{f}(x)$ for all $x \in (a,b)$. We say that $\tilde{f}$ is an extension of f.

1.2.4 (*) Weierstrass M−test

In your first year analysis courses, you looked at series. Given a sequence $(a_i)_{i=0}^{\infty}$ with $a_i \in \mathbb{R}$, we can form the partial sum:

$$S_j = \sum_{i=0}^{j} a_i.$$

If the sequence of partial sums $(S_j)_{j=0}^{\infty}$ converges, then we say that the series converges and we write:

$$\sum_{i=0}^{\infty} a_i := \lim_{j\to\infty} S_j.$$

The series is said to be *absolutely convergent* if the sum:

$$\sum_{i=0}^{\infty} |a_i|$$

converges. You may have studied various criteria for a series to converge (ratio test, root test, comparison test ...).

Often, one wishes to consider an infinite sum of *functions* rather than numbers. Let $A \subset \mathbb{R}^n$. Suppose we're given a sequence $(f_i)_{i=0}^{\infty}$ of functions $f_i : A \to \mathbb{R}^m$. If the sequence of partial sums $(S_j)_{j=0}^{\infty}$ with $S_j : A \to \mathbb{R}^m$ given by:

$$S_j(x) = \sum_{i=0}^{j} f_i(x)$$

converges (pointwise or uniformly) to some function $S : A \to \mathbb{R}^m$, then we write:

$$\sum_{i=0}^{\infty} f_i(x) := S(x),$$

with the sum converging pointwise or uniformly as appropriate.

An important and useful result for confirming that a series converges uniformly is the Weierstrass M−test.

Theorem 1.7 (Weierstrass M−test)**.** *Let $A \subset \mathbb{R}^n$. Suppose $(f_i)_{i=0}^{\infty}$ is a sequence of functions $f_i : A \to \mathbb{R}^m$. Suppose further that there exists a sequence $(M_i)_{i=0}^{\infty}$ with $M_i \in \mathbb{R}$ such that:*

i) $\sum_{i=0}^{\infty} M_i$ *converges.*

ii) $||f_i(x)|| \leq M_i$ *for all* $x \in A$, $i \in \mathbb{N}$.

Then $\sum_{i=0}^{\infty} f_i(x)$ *converges uniformly on* A.

Proof. Since $\sum_i M_i$ converges, the partial sums form a Cauchy sequence by Exercise 2.1. As a result, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$\left| \sum_{i=0}^{l} M_i - \sum_{i=0}^{m} M_i \right| = \sum_{i=m+1}^{l} M_i < \epsilon$$

for $N \leq m \leq l$. Letting $S_j(x) = \sum_{i=0}^{j} f_i(x)$, we can estimate for any $x \in A$:

$$||S_l(x) - S_m(x)|| = \left|\left| \sum_{i=m+1}^{l} f_i(x) \right|\right| \leq \sum_{i=m+1}^{l} ||f_i(x)|| \leq \sum_{i=m+1}^{l} M_i < \epsilon.$$

Thus $(S_j(x))_{j=0}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^m$. Again invoking Exercise 2.1, we have that $S_j(x) \to S(x)$ for some $S(x) \in \mathbb{R}^m$. Repeating the arguments above, we have that given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $N \leq m \leq l$ and for any $x \in A$ we have:

$$||S_l(x) - S_m(x)|| < \frac{\epsilon}{2}.$$

Sending $l \to \infty$, we conclude:

$$||S(x) - S_m(x)|| \leq \frac{\epsilon}{2} < \epsilon,$$

so that $S_j \to S$ uniformly. □

As a useful example, we can show that functions defined by power series are continuous within their radius of convergence.

Corollary 1.8. *Suppose that $(a_i)_{i=0}^{\infty}$ is a sequence of real numbers $a_i \in \mathbb{R}$, and suppose there exists $r > 0$ such that the sum*

$$\sum_{i=0}^{\infty} |a_i| \, r^i$$

converges. Then the power series:

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

converges uniformly to a continuous function on the interval $[-r, r]$.

Proof. We apply the Weierstrass $M-$test with $f_i(x) = a_i x^i$, and $M_i = |a_i| \, r^i$. □

Exercise 3.7 (*)**.** A Fourier series is a series of the form:

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx \right),$$

where $a_k, b_k \in \mathbb{R}$. Show that if $\sum_{k=1}^{\infty} (|a_k| + |b_k|)$ converges, then the Fourier series converges uniformly on $\mathbb{R}$ to a uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying:

$$f(x + 2\pi) = f(x), \quad \text{for all } x.$$

Exercise 3.8 (*)**.** Let $h : \mathbb{R} \to \mathbb{R}$ be the periodic function satisfying:

$$h(x) = |x| \qquad x \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$

and $h(x+1) = h(x)$.

a) Sketch $h(x)$.

b) Show that $h(x)$ is uniformly continuous on $\mathbb{R}$ and differentiable at each point $x \in \mathbb{R}$ such that $2x \notin \mathbb{Z}$.

Define the Takagi function[2] $T : \mathbb{R} \to \mathbb{R}$ by:

$$T(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} h\left(2^i x\right) \tag{1.4}$$

c) Show that the sum in (1.4) converges uniformly on $\mathbb{R}$ to a uniformly continuous function $T : \mathbb{R} \to \mathbb{R}$.

d) Let $x \in \mathbb{R}$. Show that there exist unique u_n, v_n such that:

  i) $2^n v_n, 2^n u_n \in \mathbb{Z}$,

  ii) $v_n - u_n = 2^{-n}$ and,

  iii) $u_n \leq x < v_n$.

e) Let $h_i(x) = \frac{1}{2^i} h\left(2^i x\right)$. Show that

  i) $h_i(u_n) = h_i(v_n) = 0$ for $i \geq n$.

  ii) $\dfrac{h_i(v_n) - h_i(u_n)}{v_n - u_n} = \pm 1$ for $i < n$.

f) Deduce that:

$$\frac{T(v_n) - T(u_n)}{v_n - u_n} = \sum_{i=0}^{n-1} \frac{h_i(v_n) - h_i(u_n)}{v_n - u_n},$$

and conclude that the sum on the right does *not* converge as $n \to \infty$.

g) Show that T is not differentiable at any x.

1.3 (*) Arzelà-Acscoli theorem

The final result we shall prove before we move on to consider integration is the Arzelà-Acscoli theorem. This can be thought of loosely as a Bolzano–Weierstrass theorem, but for sequences of functions rather than sequences of points in $\mathbb{R}^n$. It turns out that this theorem is surprisingly powerful, and is important in several branches of analysis.

Before we state the theorem, we first need a definition.

[2] Sometimes called the blancmange function.

Definition 1.5. Let $F = (f_i)_{i=0}^{\infty}$ be a sequence of continuous functions $f_i : [a, b] \to \mathbb{R}$. We say that F is *uniformly equicontinuous* if the following holds: given $\epsilon > 0$, there exists $\delta > 0$ such that:

$$|f_i(x) - f_i(y)| < \epsilon,$$

for all $i \in \mathbb{N}$ and all $x, y \in [a, b]$ with $|x - y| < \delta$.

Notice that the key point here (and the way in which this differs from uniform continuity of each f_i) is that we can choose δ *independently* of i.

Exercise 3.9 (*)**.** Suppose $F = (f_i)_{i=0}^{\infty}$ is a sequence of continuous functions $f_i : [a, b] \to \mathbb{R}$, which are differentiable on (a, b) and satisfy:

$$\sup_{x\in(a,b)} \left|f_i'(x)\right| \leq M$$

for some M independent of i. Show that F is uniformly equicontinuous.

Theorem 1.9 (Arzelà-Ascoli)**.** *Let $F = (f_i)_{i=0}^{\infty}$ be a uniformly equicontinuous sequence of functions $f_i : [a, b] \to \mathbb{R}$ satisfying:*

$$\sup_{x\in[a,b]} |f_i(x)| \leq K$$

for some K independent of i. Then F has a subsequence which converges uniformly.

Proof. First of all, note that the set $[a, b] \cap \mathbb{Q}$ is countable, so let us write $[a, b] \cap \mathbb{Q} = \{x_1, x_2, \ldots\}$. Now consider the sequence of numbers $(f_i(x_1))_{i=0}^{\infty}$. This is bounded, so by Bolzano–Weierstrass has a convergent subsequence $(f_{i_j}(x))_{j=0}^{\infty}$. To keep notation a bit easier, let's denote $f_{0,j} = f_j$ and $f_{1,j} = f_{i_j}$, so that $f_{1,j}$ is a subsequence of F with $f_{1,j}(x_1)$ convergent. Now let's consider $(f_{1,i}(x_2))_{i=0}^{\infty}$, which again is a bounded sequence of real numbers, with a convergent subsequence $f_{1,i_j}(x_2)$. Let us set $f_{2,j} = f_{1,i_j}$. This is a subsequence of $f_{1,j}$ such that $f_{2,j}(x_1)$ and $f_{2,j}(x_2)$ both converge. We can repeat this process of extracting subsequences to obtain functions $f_{k,i}$ for $k, i \in \mathbb{N}$ with the property that $(f_{k+1,i})_{i=0}^{\infty}$ is a subsequence of $(f_{k,i})_{i=0}^{\infty}$ and moreover $f_{k,i}(x_l)$ converges for $l \leq k$.

Let us define a new subsequence $G = (g_i)_{i=0}^{\infty}$ by $g_i = f_{i,i}$. This has the property that G is a subsequence of F, and moreover $g_i(x_l)$ converges for each $l \in \mathbb{N}$, let us say $g_i(x_l) \to g(x_l)$. It remains to show that in fact g_i converges uniformly. To see this, let us fix $\epsilon > 0$. Since F is equicontinuous, we may find $\delta > 0$ such that $|g_i(x) - g_i(y)| < \frac{1}{3}\epsilon$ for all $|x - y| < \delta$. Since $[a, b]$ is finite, there exist a finite number of rational points $\{x_1, \ldots x_K\} \in [a, b] \cap \mathbb{Q}$ such that $[a, b] \subset \cup_{m=1}^{K}(x_m - \delta, x_m + \delta)$. Since g_i converges at each rational point, for each $m = 1, \ldots K$ there exists N_m such that for all $i, j \geq N_m$ we have $|g_i(x_m) - g_j(x_m)| < \frac{\epsilon}{3}$. Let $N = \max\{N_1, \ldots N_K\}$ and suppose $i, j \geq N$. Fix $x \in [a, b]$. By construction, there is an $m \in \{1, \ldots, K\}$ such that $|x - x_m| < \delta$. We estimate:

$$\begin{aligned}
|g_i(x) - g_j(x)| &= |g_i(x) - g_i(x_m) + g_i(x_m) - g_j(x_m) + g_j(x_m) - g_j(x)| \\
&\leq |g_i(x) - g_i(x_m)| + |g_i(x_m) - g_j(x_m)| + |g_j(x_m) - g_j(x)| \\
&\leq \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon.
\end{aligned}$$

Thus, g_i is a Cauchy sequence in $C^0([a,b])$, so by Theorem 1.6, we deduce that $g_i \to g$ uniformly for some uniformly continuous $g : [a,b] \to \mathbb{R}$. $\square$