Let $D \subset \mathbb{C}$ be an open domain and $w : D \to \mathbb{C}$ a holomorphic function. Recall that a point $\xi \in \partial D$ is a singular point of w if w cannot be continued holomorphically to an open set including[1] ξ. Singular points include poles, essential singularities and branch points.

For a w obeying the linear equation:

$$\frac{d^2 w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0,$$

with meromorphic p, q, it is known that any singularities of w can only occur at points where p or q are singular. Regardless of the initial data $w(z_0), w'(z_0)$, the singularities are fixed.

By contrast, for some nonlinear equations singularities can occur even for holomorphic coefficients. Consider

$$\frac{dw}{dz} + z^2 = 0$$

for which the general solution is:

$$w(z) = \frac{1}{z - z_0}.$$

This has a solution at $z = z_0$ which depends on the constant of integration. It is a *moveable singularity.*

Consider a general ODE of the form:

$$\frac{d^n w}{dz^n} = F\left(\frac{d^{n-1}w}{dz^{n-1}}, \ldots, w, z\right)$$

We say such an ODE has the Painlevé property if the moveable singularities of its solutions are at worst poles. This is the case with the example above, but if we instead consider

$$\frac{dw}{dz} + z^3 = 0$$

for which the general solution is:

$$w(z) = \frac{1}{\sqrt{2(z - z_0)}},$$

the moveable singularity is a branch point, and this does not satisfy the Painleé property.

Painlevé classified all of the ODEs with this property of the form:

$$\frac{d^2 w}{dz^2} = F\left(\frac{dw}{dz}, w, z\right),$$

[1]We permit here extensions defined on some Riemann surface other than $\mathbb{C}$ so that for example $\log z$ has a singular point at $z = 0$, but not along the branch cut.

where F is a rational function. He found[2] 50 canonical types, of which 44 he could solve by functions then known (sin, cos, Jacobi elliptic functions, Bessel functions,...). The other 6 define new transcendental functions called the Painlevé transcendents.

$$(PI) \qquad \frac{d^2w}{dz^2} = 6w^2 + z$$

$$(PII) \qquad \frac{d^2w}{dz^2} = 2w^3 + zw + \alpha$$

$$(PIII) \qquad \frac{d^2w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 + \frac{1}{z}\left(-\frac{dw}{dz} + \alpha w^2 + \beta\right) + \gamma w^3 + \frac{\delta}{w}$$

$$(PIV) \qquad \frac{d^2w}{dz^2} = \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3w^3}{2} + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

$$(PV) \qquad \frac{d^2w}{dz^2} = \left(\tfrac{1}{2w} + \tfrac{1}{w-1}\right)\left(\tfrac{dw}{dz}\right)^2 - \tfrac{1}{z}\tfrac{dw}{dz} + \tfrac{(w-1)^2}{z^2}\left(\alpha w + \tfrac{\beta}{w}\right) + \tfrac{\gamma w}{z} + \tfrac{\delta w(w+1)}{w-1}$$

$$(PVI) \qquad \frac{d^2w}{dz^2} = \tfrac{1}{2}\left(\tfrac{1}{w} + \tfrac{1}{w-1} + \tfrac{1}{w-z}\right)\left(\tfrac{dw}{dz}\right)^2 - \left(\tfrac{1}{z} + \tfrac{1}{z-1} + \tfrac{1}{w-z}\right)\tfrac{dw}{dz}$$
$$+ \tfrac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \tfrac{\beta z}{w^2} + \tfrac{\gamma(z-1)}{(w-1)^2} + \tfrac{\delta z(z-1)}{(w-z)^2}\right)$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are constants.

It can be shown that if a PDE is solvable by the inverse scattering transform, then all ODE reductions of that PDE must possess the Painlevé property. This gives a necessary (but not sufficient) condition for integrability. To obtain ODE reductions of a PDE, we first identify a Lie point symmetry, and then introduce coordinates that are invariant with respect to that symmetry.

Example 1. Consider the Sine-Gordon equation in light-cone coordinates: $u_{xt} = \sin u$. This admits a Lie point symmetry:

$$\psi^s : (x, t, u) \mapsto (e^s x, e^{-s} t, u)$$

Clearly $z = xt$ is an invariant of this one-parameter group of transformations, so we seek a solution of the form $u(x, t) = F(z)$. Setting $w = e^{iF}$ and substituting into Sine-Gordon, we arrive at:

$$\frac{d^2w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{w^2}{2z} - \frac{1}{2z}.$$

which is $(PIII)$ with $\alpha = -\beta = 1/2$, $\gamma = \delta = 0$.

[2] This is a bit of historical revisionism. Painleveé made some mistakes which were corrected by later authors.