We wish to consider symmetries of some scalar PDE of order $k$ defined over $\mathbb{R}^n$. The main challenge in generalising from the ODE case is trying to keep the notation straight when there are lots of partial derivatives flying around. In order to keep notation sensible when working with high order partial derivatives it's useful to introduce *multi-index* notation. We define a multi-index $\alpha$ to be an element of $(\mathbb{Z}_{\geq 0})^n$, i.e. a $n-$vector of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$. We let $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and for $u : \mathbb{R}^n \to \mathbb{R}$ a smooth function we define

$$\frac{\partial^{|\alpha|} u}{\partial x^\alpha} := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u,$$

in other words, we differentiate $\alpha_1$ times with respect to $x_1$, $\alpha_2$ times with respect to $x_2$ and so on. We write

$$D^k u = \left(\frac{\partial^{|\alpha|} u}{\partial x^\alpha}\right)_{|\alpha| \leq k},$$

for the collection of all partial derivatives of $u$ of order at most $k$. Note this includes the derivative of order zero, which is $u$ itself. The $k^{th}$ jet space then consists of points $(x, D^k u)$ with $x \in \mathbb{R}^n$, and a $k^{th}$ order PDE is a relation of the form:

$$\Delta(x, D^k u) = 0.$$

Suppose that $\psi^s : (x, u) \mapsto (\tilde{x}, \tilde{u})$ is a smooth one-parameter group of transformations. By the chain rule, this induces a smooth one-parameter group of transformations on the $k-th$ jet space by:

$$Pr^{(k)} \psi^s : (x, D^k u) \mapsto \left(\tilde{x}, \widetilde{D^k u}\right)$$

where the components of $\widetilde{D^k u}$ are:

$$\widetilde{D^k u} = \left(\frac{\partial^{|\alpha|} \tilde{u}}{\partial \tilde{x}^\alpha}\right)_{|\alpha| \leq k}.$$

Since $\psi^s$ is a smooth one-parameter group of transformations, it is generated by a vector field $V$ which we can write as:

$$V = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}.$$

The prolongation $Pr^{(k)} \psi^s$ will be generated by the prolongation of $V$, which will take the form:

$$Pr^{(k)} V = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} + \sum_{|\alpha| \leq k} \eta^\alpha \frac{\partial}{\partial u^{(\alpha)}}$$

here $\eta^{(\alpha)}$ are undetermined functions, except for $\eta^{(0,\dots,0)} = \eta$ and we have introduced the notation:

$$u^{(\alpha)} = \frac{\partial^{|\alpha|} u}{\partial x^\alpha}.$$

As in the one-dimensional case, we can construct the prolongation of $V$ *without* having to first construct the prolongation of $\psi^s$. We do this order by order in $k$. Let us first consider finding $\eta^\alpha$ for $|\alpha| = 1$, so that $\alpha = e_j$ for some $j = 1, \dots, n$. Then $u^{(\alpha)} = \frac{\partial u}{\partial x^j}$. We make use of the contact condition:

$$d\tilde{u} = \sum_{i=1}^{n} \frac{\partial \tilde{u}}{\partial \tilde{x}^i} d\tilde{x}^i.$$

Using $\tilde{u} = u + s\eta(x, u) + o(s)$ for small $s$, the left hand side can be written:

$$d\tilde{u} = du + s \sum_{l=1}^{n} (D_{x^l}\eta)\, dx^l + o(s) = \sum_{l=1}^{n} \left( \frac{\partial u}{\partial x^l} + sD_{x^l}\eta \right) dx^l + o(s)$$

where $D_{x^l}$ is the total derivative operator in the $l$ direction:

$$\begin{aligned} D_{x^l} &= \frac{\partial}{\partial x^l} + \frac{\partial u}{\partial x_l}\frac{\partial}{\partial u} + \sum_{k=1}^{n} \frac{\partial(\partial_k u)}{\partial x^l} \frac{\partial}{\partial(\partial_k u)} + \dots \\ &= \frac{\partial}{\partial x^l} + \sum_{\alpha} u^{(\alpha+e_l)} \frac{\partial}{\partial u^{(\alpha+e_l)}} \end{aligned}$$

Since $\eta$ is a function only of $x, u$, we have:

$$D_{x^l}\eta = \frac{\partial \eta}{\partial x^l} + \frac{\partial u}{\partial x^l}\frac{\partial \eta}{\partial u}.$$

Now, since $\tilde{x}^i = x^i + s\xi^i(x, u) + o(s)$, we have:

$$d\tilde{x}^i = dx^i + s \sum_{m=1}^{n} (D_{x^m}\xi^i) dx^m + o(s) = \sum_{m=1}^{n} \left( \delta_m{}^i + sD_{x^m}\xi^i \right) dx^m + o(s)$$

Putting these expressions together, we have:

$$\sum_{l=1}^{n} \left( \frac{\partial u}{\partial x^l} + sD_{x^l}\eta \right) dx^l = \sum_{i=1}^{n}\sum_{m=1}^{n} \frac{\partial \tilde{u}}{\partial \tilde{x}^i} \left( \delta_m{}^i + sD_{x^m}\xi^i \right) dx^m + o(s)$$

so that we have:

$$\left( \frac{\partial u}{\partial x^l} + sD_{x^l}\eta \right) = \sum_{i=1}^{n} \frac{\partial \tilde{u}}{\partial \tilde{x}^i} \left( \delta_l{}^i + sD_{x^l}\xi^i \right) + o(s)$$

Multiplying both sides by $\left( \delta_j{}^l - sD_{x^j}\xi^l \right)$ and summing over $l$, we have:

$$\sum_{l=1}^{n} \left( \frac{\partial u}{\partial x^l} + sD_{x^l}\eta \right) \left( \delta_j{}^l - sD_{x^j}\xi^l \right) = \sum_{l=1}^{n}\sum_{i=1}^{n} \frac{\partial \tilde{u}}{\partial \tilde{x}^i} \left( \delta_l{}^i + sD_{x^l}\xi^i \right) \left( \delta_j{}^l - sD_{x^j}\xi^l \right) + o(s)$$

Expanding, we conclude:

$$\frac{\partial \tilde{u}}{\partial \tilde{x}^j} = \frac{\partial u}{\partial x^j} + s\left( D_{x^j}\eta - \sum_{l=1}^{n} \frac{\partial u}{\partial x^l} D_{x^j}\xi^l \right) + o(s)$$

So that we can finally read off:

$$\eta^{e_j} = D_{x^j}\eta - \sum_{l=1}^{n} u^{(e_l)} D_{x^j}\xi^l,$$

which is consistent with our result in the one-dimensional case. This permits us to find all of the components of $Pr^{(1)}V$, since:

$$Pr^{(1)}V = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \sum_{j=1}^{n} \eta^{e_j} \frac{\partial}{\partial u^{(e_j)}}$$

We can now iterate this result. Using the contact condition:

$$d\widetilde{u^{(\alpha)}} = \sum_{i=1}^{n} \frac{\partial \widetilde{u^{(\alpha)}}}{\partial \tilde{x}^i} d\tilde{x}^i$$

together with the fact that $\widetilde{u^{(\alpha)}} = u^{(\alpha)} + s\eta^\alpha + o(s)$, we can work through the computations above again to determine:

$$\eta^{\alpha+e_j} = D_{x^j}\eta^\alpha - \sum_{l=1}^{n} u^{(\alpha+e_l)} D_{x^j}\xi^l.$$

In this way we can find arbitrary prolongations of $V$.

**Example 1.** Suppose we consider a PDE defined on $\mathbb{R}^2$ with coordinates $(x^1, x^2) = (x, y)$. Consider the vector field:

$$V = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}.$$

so that $\xi^1 = y$, $\xi^2 = x$ and $\eta = u$. Then:

$$\eta^{(1,0)} = D_{e_1}\eta - u_x D_{e_1}\xi^1 - u_y D_{e_1}\xi^2 = u_x - u_x \cdot 0 - u_y \cdot 1 = u_x - u_y$$

and

$$\eta^{(0,1)} = D_{e_2}\eta - u_x D_{e_2}\xi^1 - u_y D_{e_2}\xi^2 = u_y - u_x \cdot 1 - u_y \cdot 0 = u_y - u_x$$

so that:

$$Pr^{(1)}V = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + (u_x - u_y)\frac{\partial}{\partial u_x} + (u_y - u_x)\frac{\partial}{\partial u_y}$$

Continuing, we have:

$$\eta^{(2,0)} = D_{e_1}\eta^{(1,0)} - u_{xx} D_{e_1}\xi^1 - u_{xy} D_{e_1}\xi^2 = u_{xx} - u_{xy} - u_{xy}$$

$$\eta^{(1,1)} = D_{e_1}\eta^{(0,1)} - u_{yx}D_{e_1}\xi^1 - u_{yy}D_{e_1}\xi^2 = u_{xy} - u_{xx} - u_{yy}$$
$$\eta^{(1,1)} = D_{e_2}\eta^{(1,0)} - u_{xx}D_{e_2}\xi^1 - u_{xy}D_{e_2}\xi^2 = u_{xy} - u_{yy} - u_{xx}$$
$$\eta^{(0,2)} = D_{e_2}\eta^{(0,1)} - u_{yx}D_{e_2}\xi^1 - u_{yy}D_{e_2}\xi^2 = u_{yy} - u_{xy} - u_{xy}$$

Here we've verified that for $\eta^{(1,1)}$ we get the same result regardless of how we compute it. Thus we have:

$$Pr^{(2)}V = Pr^{(1)}V + (u_{xx} - 2u_{xy})\frac{\partial}{\partial u_{xx}} + 2\,(u_{xy} - u_{xx} - u_{yy})\frac{\partial}{\partial u_{xy}} + (u_{yy} - 2u_{xy})\frac{\partial}{\partial u_{yy}}$$

Consider the wave equation, defined by:

$$\Delta(x, D^2u) = u_{xx} - u_{yy}$$

We have:

$$Pr^{(2)}V(\Delta) = (u_{xx} - 2u_{xy}) - (u_{yy} - 2u_{xy}) = u_{xx} - u_{yy} = \Delta,$$

so if $\Delta = 0$ then $Pr^{(2)}V(\Delta)$, so we conclude that $V$ generates a solution of the wave equation.