**1 Analysis of PDE Sample Question**

We work over $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

a) State the Gagliardo–Nirenberg–Sobolev inequality.

b) Let $1 \leqslant p < n$ and set $p_* = np/(n-p)$. Show that it $q \neq p_*$, then there can exist no constant $C$ such that

$$\|u\|_{L^q} \leqslant C\|Du\|_{L^p}$$

holds for all $u \in C_c^\infty(\mathbb{R}^n)$.

*Hint: you may find it useful to consider the family of functions* $\phi_\lambda(x) := \phi(\lambda x)$ *for some fixed non-zero* $\phi \in C_c^\infty(\mathbb{R}^n)$ *and* $\lambda > 0$.

c) Suppose for some $1 \leqslant p, q, < \infty$ and $a, b \in \mathbb{R}$ there exists a constant $C$ such that the inequality:

$$\|u|x|^a\|_{L^q} \leqslant C \left\| |Du||x|^b \right\|_{L^p} \qquad (\dagger)$$

holds for all $u \in C_c^\infty(\mathbb{R}^n)$. By arguing as in part *b*), or otherwise, find a relation between $p, q, a, b, n$.

d) Show that if $p = q$, $b = a + 1$ and $ap > -n$, there indeed exists a constant $C$ such that ($\dagger$) holds for all $u \in C_c^\infty(\mathbb{R}^n)$.

*Hint: note that* $|x|^{ap} = \frac{1}{n+ap} div\ (x|x|^{ap})$, *and integrate by parts.*

a) *The Gagliardo–Nirenberg–Sobolev inequality states that if* $1 \leqslant p < n$ *and* $p_* = np/(n-p)$*, then there exists a constant* $C$ *such that:*

$$\|u\|_{L^{p_*}} \leqslant C\|Du\|_{L^p}$$

*holds for all* $u \in C_c^\infty(\mathbb{R}^n)$ *(or* $u \in W^{1,p}(\mathbb{R}^n)$*)*

b) *Following the hint, we compute:*

$$\|\phi_\lambda\|_{L^q} = \left(\int_{\mathbb{R}^n} |\phi(\lambda x)|^q dx\right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}^n} |\phi(y)|^q \lambda^{-n} dy\right)^{\frac{1}{q}} = \lambda^{-\frac{n}{q}}\|\phi\|_{L^q}$$

*Similarly,*

$$\|D\phi_\lambda\|_{L^p} = \left(\int_{\mathbb{R}^n} |\lambda D\phi(\lambda x)|^p dx\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^n} |D\phi(y)|^p \lambda^{p-n} dy\right)^{\frac{1}{p}} = \lambda^{1-\frac{n}{p}}\|\phi\|_{L^q}$$

*Now applying the inequality to* $u = \phi_\lambda$ *and rearranging, we conclude:*

$$\lambda^{\frac{n}{p}-\frac{n}{q}-1} \leqslant C\frac{\|D\phi\|_{L^p}}{\|\phi\|_{L^q}}$$

*We arrive at a contradiction by sending* $\lambda$ *to* $0$ *or* $\infty$*, unless* $\frac{n}{p} - \frac{n}{q} - 1 = 0$*, which occurs precisely when* $q = p_*$*.*

c) *Arguing as in the previous part, we note that:*

$$\|\phi_\lambda |x|^a\|_{L^q} = \left(\int_{\mathbb{R}^n} |\phi_\lambda|^q |x|^{qa} dx\right)^{\frac{1}{q}} = \lambda^{-\frac{n+aq}{q}} \left(\int_{\mathbb{R}^n} |\phi|^q |x|^a dx\right)^{\frac{1}{q}} = \lambda^{-\frac{n}{q}-a}\|\phi|x|^a\|_{L^q}$$

*and*

$$\left\|D\phi_\lambda |x|^b\right\|_{L^p} = \left(\int_{\mathbb{R}^n} |D\phi_\lambda|^p |x|^{pb} dx\right)^{\frac{1}{p}} = \lambda^{1-\frac{n+pb}{p}} \left(\int_{\mathbb{R}^n} |D\phi|^p |x|^b dx\right)^{\frac{1}{p}} = \lambda^{1-\frac{n}{p}-b}\left\|D\phi|x|^b\right\|_{L^p}$$

*We see that, in order for the inequality (†) to hold for all* $u \in C_c^\infty(\mathbb{R}^n)$ *we must have:*

$$\frac{n}{p} + b = \frac{n}{q} + a + 1.$$

d) *Following the hint, we compute:*

$$\int_{\mathbb{R}^n} |\phi|^p |x|^{ap} dx = \frac{1}{n+ap}\int_{\mathbb{R}^n} |\phi|^p \text{div } (x|x|^{ap}) dx = -\frac{p}{n+ap}\int_{\mathbb{R}^n} (\text{sgn }\phi) x \cdot D\phi |\phi|^{p-1}|x|^{ap} dx$$

*The condition* $ap > -n$ *ensures that there is no contribution from the origin. Applying Hölder's inequality, we can estimate the integral on the RHS by:*

$$\left| -\frac{p}{n+ap}\int_{\mathbb{R}^n} (\text{sgn }\phi) x \cdot D\phi |\phi|^{p-1}|x|^{ap} dx\right| \leqslant \frac{p}{n+ap}\|\phi^{p-1}|x|^{a(p-1)}\|_{L^{\frac{p}{p-1}}} \left\||D\phi||x|^{1+a}\right\|_{L^p}$$

*We deduce:*

$$\|\phi|x|^a\|^p_{L^p} \leqslant \frac{p}{n+ap}\|\phi|x|^a\|^{p-1}_{L^p}\left\||D\phi||x|^{1+a}\right\|_{L^p}$$

*and the result follows on cancelling* $\|\phi|x|^a\|^{p-1}_{L^p}$ *from both sides.*