Course materials at : www.dpmms.cam.ac.uk/~cmw50

1 Analysis of PDE Sample Question

We work over $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

a) State the Gagliardo–Nirenberg–Sobolev inequality. | bookwork

b) Let $1 \leqslant p < n$ and set $p_* = np/(n-p)$. Show that it $q \neq p_*$, then there can exist no constant C such that
$$\|u\|_{L^q} \leqslant C\|Du\|_{L^p}$$
holds for all $u \in C_c^\infty(\mathbb{R}^n)$. | Example sheet.

Hint: you may find it useful to consider the family of functions $\phi_\lambda(x) := \phi(\lambda x)$ for some fixed non-zero $\phi \in C_c^\infty(\mathbb{R}^n)$ and $\lambda > 0$.

c) Suppose for some $1 \leqslant p, q, < \infty$ and $a, b \in \mathbb{R}$ there exists a constant C such that the inequality:
$$\|u|x|^a\|_{L^q} \leqslant C \left\| |Du||x|^b \right\|_{L^p} \qquad (\dagger)$$
holds for all $u \in C_c^\infty(\mathbb{R}^n)$. By arguing as in part b), or otherwise, find a relation between p, q, a, b, n. | new, similar to b).

d) Show that if $p = q$, $b = a + 1$ and $ap > -n$, there indeed exists a constant C such that ($\dagger$) holds for all $u \in C_c^\infty(\mathbb{R}^n)$.

Hint: note that $|x|^{ap} = \frac{1}{n+ap} div\ (x|x|^{ap})$, and integrate by parts.

1a) Gagliardo-Nirenberg-Sobolev

If $1 \leq p < n$, $p_* = \dfrac{np}{n-p}$ then $\exists$ constant C s.t.

$$\|u\|_{L^{p_*}} \leq C\|Du\|_{L^p} \qquad \forall\ u \in C_c^\infty(\mathbb{R}^n)$$

in particular, $W^{1,p}(\mathbb{R}^n) \subset L^{p_*}(\mathbb{R}^n)$ with this estimate.

b) $\phi \in C_c^\infty(\mathbb{R}^n)$, non-zero, $\phi_\lambda(x) := \phi(\lambda x)$ $\quad \lambda > 0$

$$\|\phi_\lambda\|_{L^q} = \left(\int_{\mathbb{R}^n} |\phi(\lambda x)|^q\, dx\right)^{1/q} \qquad \begin{aligned} y &= \lambda x \\ dx &= \lambda^{-n} dy \end{aligned}$$

$$= \left(\int_{\mathbb{R}^n} |\phi(y)|^q \lambda^{-n}\, dx\right)^{1/q}$$

$$= \lambda^{-n/q} \|\phi\|_{L^q}.$$

$$\|D\phi_\lambda\|_{L^p} = \left(\int_{\mathbb{R}^n} |D\phi_\lambda(x)|^p\,dx\right)^{1/p} = \left(\int_{\mathbb{R}^n} |\lambda D\phi(\lambda x)|^p\,dx\right)^{1/p}$$

$$= \left(\int_{\mathbb{R}^n} |D\phi(y)|^p \lambda^{p-n}\,dy\right)^{1/p}$$

$$= \lambda^{1-n/p}\|D\phi\|_{L^p}.$$

Suppose $\exists\, C$ s.t.

$$\|u\|_{L^q} \leq C\|Du\|_{L^p} \qquad \forall u \in C_c^\infty(\mathbb{R}^n)$$

set $u = \phi_\lambda$

$$\lambda^{-n/q}\|\phi\|_{L^q} \leq C\cdot\lambda^{1-n/p}\|D\phi\|_{L^p}$$

$\Rightarrow$

$$\lambda^{\frac{n}{p}-\frac{n}{q}-1} \leq C\frac{\|D\phi\|_{L^p}}{\|\phi\|_{L^q}}$$

If $\frac{n}{p}-\frac{n}{q}-1 \neq 0$, then either $\lambda\to 0$ or $\lambda\to\infty$ sends

$$\lambda^{n/p-n/q-1} \to \infty,$$ gives contradiction.

$\therefore$ Result cannot hold unless $\frac{n}{p}-1 = \frac{n}{q} \Rightarrow q = \frac{np}{n-p} = p_*$.

c) If ϕ_λ as above

$$\|\phi_\lambda |x|^a\|_{L^q} = \left(\int_{\mathbb{R}^n} |\phi(\lambda x)|^q |x|^{aq}\,dx\right)^{1/q}$$

$y = \lambda x$
$dx = \lambda^{-n}dy$

$$= \left(\int_{\mathbb{R}^n} |\phi(y)|^q |y|^{aq}\cdot\lambda^{-n-aq}\,dy\right)^{1/q}$$

$$= \lambda^{-n/q-a}\|\phi|x|^a\|_{L^q}$$

$$\|D\phi_\lambda |x|^b\|_{L^p} = \left(\int_{\mathbb{R}^n} |\lambda D\phi(\lambda x)|^p \, |x|^{bp} \, dx \right)^{1/p}$$

$$= \left(\int_{\mathbb{R}^n} |D\phi(y)|^p |y|^{bp} \cdot \lambda^{p-n-bp} \, dy \right)^{1/p}$$

$$= \lambda^{1-n/p-b}$$

Suppose $\exists\, C$ s.t. $\quad \|u|x|^a\|_{L^q} \le C\|Du|x|^b\|_{L^p} \quad \forall u \in C_c^\infty(\mathbb{R}^n)$

$$\Rightarrow \quad \lambda^{-n/q-a-1+\frac{n}{p}+b} \le C \, \frac{\|D\phi|x|^b\|_{L^p}}{\|\phi|x|^a\|_{L^q}}$$

as above, must have

$$\underline{\frac{n}{p}+b = \frac{n}{q}+a+1.} \qquad \text{(otherwise contradiction)}$$

d) $p=q$, $b=a+1$

$$\|u|x|^a\|_{L^p}^p = \int_{\mathbb{R}^n} |u(x)|^p \cdot |x|^{ap} \, dx$$

$$= \int_{\mathbb{R}^n} |u(x)|^p \cdot \frac{1}{n+ap} \, \mathrm{div}\left(x|x|^{ap}\right) dx$$

see below.

$$= \frac{-1}{n+ap} \int_{\mathbb{R}^n} |x|^{ap} \, x \cdot D(|u|^p) \, dx$$

$$= \frac{-p}{n+ap} \int_{\mathbb{R}^n} |x|^{ap} \, \mathrm{sign}(u) \, x \cdot Du \, |u|^{p-1} \, dx$$

By Hölder

$$\left| \frac{p}{n+ap} \int_{\mathbb{R}^n} |x|^{ap} \operatorname{sign}(u)\, x \cdot \underline{Du}\, |u|^{p-1}\, dx \right|$$

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad q = \frac{p}{p-1}$$

$$\leq \frac{p}{n+ap} \| Du\, |x|^{a+1} \|_{L^p} \underbrace{\| |u|^{p-1} |x|^{ap-a} \|_{L^{\frac{p}{p-1}}}}_{\left(\int \left(|u|^{p-1} |x|^{a(p-1)}\right)^{p/p-1} dx \right)^{\frac{p-1}{p}}}$$

$$= \| u\, |x|^a \|_{L^p}^{p-1}$$

ie

$$\| u\, |x|^a \|_{L^p}^{p} \leq \frac{p}{n+ap} \| Du\, |x|^{a+1} \|_{L^p} \| u |x|^a \|_{L^p}^{p-1}$$

cancelling $\| u |x|^a \|_{L^p}^{p-1}$, result follows $\left(C = \frac{p}{n+ap} \right)$.

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} |u|^q \operatorname{div}(x |x|^{ap})\, dx = - \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} |x|^{ap} x \cdot D |u|^q\, dx$$

$$- \underbrace{\int_{\partial B_\varepsilon(0)} \underbrace{|u^q|}_{\leq C} \underbrace{x |x|^{ap} \cdot \hat{x}}_{\varepsilon^{1+ap}} \underbrace{dS}_{\varepsilon^{n-1}}}_{\leq C \cdot \varepsilon^{n+ap}} = o(1) \text{ as } \varepsilon \to 0.$$

Sending ε to zero justifies the integration by parts.