EXISTENCE OF MINIMAL MODELS FOR VARIETIES
OF LOG GENERAL TYPE

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Abstract. We prove that the canonical ring of a smooth projective variety is finitely generated.

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1. Introduction

The purpose of this paper is to prove the following result in birational algebraic geometry:

**Theorem 1.1.** Let \((X, \Delta)\) be a projective kawamata log terminal pair. If \(\Delta\) is big and \(K_X + \Delta\) is pseudo-effective then \(K_X + \Delta\) has a log terminal model.

In particular, it follows that if \(K_X + \Delta\) is big then it has a log canonical model and the canonical ring is finitely generated. We will prove that it then follows that if \(X\) is a smooth projective variety, then the ring

\[
R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)),
\]

is finitely generated.

The birational classification of complex projective surfaces was understood by the Italian Algebraic Geometers in the early 20th century: If \(X\) is a smooth complex projective surface of nonnegative Kodaira dimension, that is \(\kappa(X, K_X) \geq 0\), then there is a unique smooth surface \(Y\) birational to \(X\) such that the canonical class \(K_Y\) is nef (that is \(K_Y \cdot C \geq 0\) for any curve \(C \subset Y\)). \(Y\) is obtained from \(X\) simply by contracting all \(-1\)-curves, that is all smooth rational curves \(E\) with \(K_X \cdot E = -1\). If, on the other hand, \(\kappa(X, K_X) = -\infty\), then \(X\) is birational to either \(\mathbb{P}^2\) or a ruled surface over a curve of genus \(g > 0\).

The Minimal Model Program aims to generalise the classification of complex projective surfaces to higher dimensional varieties. The main goal of this program is to show that given any \(n\)-dimensional smooth complex projective variety \(X\), we have:

- If \(\kappa(X, K_X) \geq 0\), then there exists a **minimal model**, that is a variety \(Y\) birational to \(X\) such that \(K_Y\) is nef.
• If \( \kappa(X, K_X) = -\infty \), then there is a variety \( Y \) birational to \( X \) which admits a \textbf{Fano fibration}, that is a morphism \( Y \to Z \) whose general fibres \( F \) have ample anticanonical class \(-K_F\).

It is possible to exhibit 3-folds which have no smooth minimal model, see for example (16.17) of [13], and so one must allow varieties \( X \) with singularities. However, these singularities cannot be arbitrary. At the very minimum, we must still be able to compute \( K_X \cdot C \) for any curve \( C \subset X \). So, we insist that \( K_X \) is \( \mathbb{Q} \)-Cartier (or sometimes we require the stronger property that \( X \) is \( \mathbb{Q} \)-factorial). We also require that \( X \) and the minimal model \( Y \) have the same pluricanonical forms. This condition is essentially equivalent to requiring that the induced birational map \( \phi: X \dashrightarrow Y \) is \( K_X \)-negative.

There are two natural ways to construct the minimal model (it turns out that if one can construct a minimal model for pseudo-effective \( K_X \), then we can construct Mori fibre spaces whenever \( K_X \) is not pseudo-effective). Since one of the main ideas of this paper is to blend the techniques of both methods, we describe both methods.

The first method is to use the ideas behind finite generation. If the canonical ring

\[
R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)),
\]

is finitely generated and \( K_X \) is big, then the canonical model \( Y \) is nothing more than the Proj of \( R(X, K_X) \). It is then automatic that the induced rational map \( \phi: X \dashrightarrow Y \) is \( K_X \)-negative.

The other natural way to ensure that \( \phi \) is \( K_X \)-negative is to factor \( \phi \) into a sequence of elementary steps all of which are \( K_X \)-negative. We now explain one way to achieve this factorisation:

If \( K_X \) is not nef, then, by the Cone Theorem, there is a rational curve \( C \subset X \) such that \( K_X \cdot C < 0 \) and a morphism \( f: X \to Z \) which is surjective, with connected fibres, on to a normal projective variety and which contracts an irreducible curve \( D \) if and only if \([D] \in \mathbb{R}^+[C] \subset N_1(X)\). Note that \( \rho(X/Z) = 1 \) and \(-K_X \) is \( f \)-ample. We have the following possibilities:

• If \( \dim Z < \dim X \), this is the required Fano fibration.
• If \( \dim Z = \dim X \) and \( f \) contracts a divisor, then we say that \( f \) is a divisorial contraction and we replace \( X \) by \( Z \).
• If \( \dim Z = \dim X \) and \( f \) does not contract a divisor, then we say that \( f \) is a small contraction. In this case \( K_Z \) is not \( \mathbb{Q} \)-Cartier, so that we cannot replace \( X \) by \( Z \). Instead, we would like to replace \( f: X \to Z \) by its flip \( f^+: X^+ \to Z \) where \( X^+ \) is isomorphic to \( X \) in codimension 1 and \( K_{X^+} \) is \( f^+ \)-ample. In
other words, we wish to replace some $K_X$-negative curves by $K_X^+$-positive curves.

The idea is to simply repeat the above procedure until we obtain either a minimal model or a Fano fibration. For this procedure to succeed, we must show that flips always exist and that they eventually terminate. Since the Picard number $\rho(X)$ drops by one after each divisorial contraction, there can be at most finitely many divisorial contractions. So we must show that there is no infinite sequence of flips.

This program was successfully completed for 3-folds in the 1980’s by the work of Kawamata, Kollár, Mori, Reid, Shokurov and others. In particular, the existence of 3-fold flips was proved by Mori in [32].

Naturally, one would hope to extend these results to dimension 4 and higher proceeding by induction on the dimension.

Recently, Shokurov has shown the existence of flips in dimension 4 [40] and Hacon and McKernan [11] have shown that assuming the minimal model program in dimension $n - 1$ (or even better simply finiteness of minimal models in dimension $n - 1$), then flips exist in dimension $n$. Thus we get an inductive approach to finite generation.

Unfortunately the problem of showing termination of an arbitrary sequence of flips seems to be a very difficult problem and in dimension $\geq 4$ only some partial answers are available. Kawamata, Matsuda and Matsuki proved [23] the termination of terminal 4-fold flips, Matsuki has shown [31] the termination of terminal 4-fold flops and Fujino has shown [6] the termination of canonical 4-fold (log) flips. Alexeev, Hacon and Kawamata [2] have shown the termination of Kawamata log terminal 4-fold flips when $-(K_X + \Delta)$ is effective and the existence of minimal models of Kawamata log terminal 4-folds when either $\Delta$ or $K_X + \Delta$ is big by showing the termination of a certain sequence of flips (those that appear in the MMP with scaling). However, it is known that termination of flips follows from two natural conjectures on the behaviour of the log discrepancies of $n$-dimensional pairs (namely the ascending chain condition for minimal log discrepancies and semicontinuity of log discrepancies, cf. [41]). Moreover, if $\kappa(X, K_X + \Delta) \geq 0$, Birkar has shown [3] that it suffices to establish acc for log canonical thresholds and the MMP in dimension one less.

We now turn to the main result of the paper:

**Theorem 1.2.** Let $(X, \Delta)$ be a Kawamata log terminal pair, where $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi : X \rightarrow U$ be a proper morphism of quasi-projective varieties.

If either $\Delta$ is $\pi$-big and $K_X + \Delta$ is $\pi$-pseudo-effective or $K_X + \Delta$ is $\pi$-big, then
(1) $K_X + \Delta$ has a log terminal model over $U$,
(2) if $K_X + \Delta$ is $\pi$-big then $K_X + \Delta$ has a log canonical model over $U$, and
(3) if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then the $\mathcal{O}_U$-algebra

$$\bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor),$$

is finitely generated.

We now present some consequences of (1.2), most of which are known to follow from the MMP. Even though we do not prove termination of flips, we are able to derive many of the consequences of the existence of the MMP. In many cases we do not state the strongest results possible; anyone interested in further applications is directed to the references. We group these consequences under different headings.

1.1. Minimal models. An immediate consequence of (1.2) is:

**Corollary 1.1.1.** Let $X$ be a smooth projective variety of general type. Then
(1) $X$ has a minimal model,
(2) $X$ has a canonical model,
(3) the ring

$$\bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)),$$

is finitely generated, and
(4) $X$ has a model with a K"ahler-Einstein metric.

Note that (4) follows from (2) and Theorem D of [5]. Note that Siu has announced a proof of finite generation for varieties of general type, using analytic methods, see [42].

**Corollary 1.1.2.** Let $(X, \Delta)$ be a kawamata log terminal pair, where $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Then the ring

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor),$$

is finitely generated.

Let us emphasize that in (1.1.2) we make no assumption about $K_X + \Delta$ or $\Delta$ being big. Indeed Fujino and Mori, [8], proved that (1.1.2) follows from the case when $K_X + \Delta$ is big.

We will now turn our attention to the geography of minimal models. It is well known that log terminal models are not unique. The first
natural question about log terminal models is to understand how any two are related. In fact there is a very simple connection:

**Corollary 1.1.3.** Let \( \pi: X \to U \) be a projective morphism of normal quasi-projective varieties. Suppose that \( K_X + \Delta \) is kawamata log terminal and \( \Delta \) is big over \( U \). Let \( \phi_i: X \to Y_i \), \( i = 1 \) and \( 2 \) be two log terminal models of \((X, \Delta)\) over \( U \). Let \( \Gamma_i = \phi_i^* \Delta \).

Then the birational map \( Y_1 \to Y_2 \) is the composition of a sequence of \((K_{Y_1} + \Gamma_1)\)-flops and log crepant extractions over \( U \).

Note that (1.1.3) has been generalised recently to the case when \( \Delta \) is not assumed big, [19]. The next natural problem is to understand how many different models there are. Even if log terminal models are not unique, there are only finitely many. In fact Shokurov realised that much more is true. He realised that the dependence on \( \Delta \) is well-behaved.

To explain this, we need some definitions:

**Definition 1.1.4.** Let \( \pi: X \to U \) be a projective morphism of normal quasi-projective varieties, and let \( V \) be a finite dimensional affine subspace of the real vector space of Weil divisors on \( X \) which is defined over the rationals. Define

\[
L = \{ \Delta \in V \mid K_X + \Delta \text{ is log canonical} \},
\]
\[
N = \{ \Delta \in L \mid K_X + \Delta \text{ is nef over } U \}.
\]

Moreover, fixing an \( \mathbb{R} \)-divisor \( A \), whose support has no common components with any element of \( V \), define

\[
V_A = \{ \Delta \mid \Delta = A + B, B \in V \},
\]
\[
L_A = \{ \Delta \in V_A \mid K_X + \Delta \text{ is log canonical} \},
\]
\[
P_A = \{ \Delta \in L_A \mid K_X + \Delta \text{ is pseudo-effective over } U \},
\]
\[
P_A' = \{ \Delta \in L_A \mid K_X + \Delta \text{ is nef over } U \}.
\]

Given a birational map \( \phi: X \to Y \) over \( U \), whose inverse does not contract any divisors, define

\[
P_Y = \{ \Delta \in P_A \mid (Y, \Gamma = \phi_* \Delta) \text{ is a weak log canonical model for } (X, \Delta) \text{ over } U \},
\]
and given a rational map \( \psi: X \to Z \) over \( U \), define

\[
Q_Z = \{ \Delta \in P_A \mid Z \text{ is an ample model for } (X, \Delta) \text{ over } U \},
\]
(cf. (3.5.4) for the definitions of weak log canonical model and ample model for \((X, \Delta)\) over \( U \)).

In nearly all applications, \( A \) will be an ample \( \mathbb{Q} \)-divisor over \( U \). In this case, we often assume that \( A \) is general in the sense that we fix a positive integer such that \( kA \) is very ample, and we assume that
$A = \frac{1}{k}A'$, where $A' \in |kA|$ is very general. With this choice of $A$, we have
\[ \mathcal{N}_A \subset \mathcal{P}_A \subset \mathcal{L}_A = \mathcal{L} + A \subset V_A = V + A, \]
and the condition that the support of $A$ has no common components with any element of $V$ is then automatic. The following result was first proved by Shokurov [38] assuming the existence and termination of flips:

**Corollary 1.1.5.** Let $\pi : X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $V$ be a finite dimensional affine subspace of the real vector space of Weil divisors which is defined over the rationals. Suppose there is a divisor $\Delta_0 \in V$ such that $K_X + \Delta_0$ is Kawamata log terminal. Let $A$ be an ample $\mathbb{Q}$-divisor over $U$, which has no components in common with any element of $V$.

1. There are finitely many birational maps $\phi_i : X \longrightarrow Y_i$, $1 \leq i \leq p$, whose inverses do not contract any divisors, such that
\[ \mathcal{P}_A = \bigcup_{i=1}^{p} \mathcal{P}_i, \]
where each $\mathcal{P}_i = \mathcal{P}_{Y_i}$ is a rational polytope. Moreover, if $\phi : X \longrightarrow Y$ is a log terminal model of $(X, \Delta)$, for some $\Delta \in \mathcal{P}_A$, then $\phi = \phi_i$, for some $1 \leq i \leq p$.
2. There are finitely many rational maps $\psi_j : X \longrightarrow Z_j$, $1 \leq j \leq q$ which partition $\mathcal{P}_A$ into the subsets $\mathcal{Q}_j = \mathcal{Q}_{Z_j}$.
3. For every $1 \leq i \leq p$ there is a $1 \leq j \leq q$ and a morphism $f_{ij} : Y_i \longrightarrow Z_j$ such that $\mathcal{P}_i \subset \mathcal{Q}_j$.

In particular $\mathcal{P}_A$ and each $\mathcal{Q}_j$ are rational polytopes.

**Definition 1.1.6.** Let $(X, \Delta)$ be a Kawamata log terminal pair and let $D$ be a big divisor. Suppose that $K_X + \Delta$ is not pseudo-effective. The effective log threshold is
\[ \sigma(X, \Delta, D) = \sup \{ t \in \mathbb{R} \mid D + t(K_X + \Delta) \text{ is pseudo-effective} \}. \]

The **Kodaira energy** is the reciprocal of the effective log threshold.

Following ideas of Batyrev, one can easily show that:

**Corollary 1.1.7.** Let $(X, \Delta)$ be a Kawamata log terminal pair and let $D$ be an ample divisor. Suppose that $K_X + \Delta$ is not pseudo-effective.

If both $K_X + \Delta$ and $D$ are $\mathbb{Q}$-Cartier then the effective log threshold and the Kodaira energy are rational.
Definition 1.1.8. Let $\pi: X \rightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $D^\bullet = (D_1, D_2, \ldots, D_k)$ be a sequence of $\mathbb{Q}$-divisors on $X$. The sheaf of $\mathcal{O}_U$-modules

$$R(\pi, D^\bullet) = \bigoplus_{m \in \mathbb{N}^k} \pi_* \mathcal{O}_X(\sum m_i D_{i,i}),$$

is called the Cox ring associated to $D^\bullet$.

Using (1.1.5) one can show that adjoint Cox rings are finitely generated:

Corollary 1.1.9. Let $\pi: X \rightarrow U$ be a projective morphism of normal quasi-projective varieties. Fix $A$ an ample $\mathbb{Q}$-divisor over $U$. Let $\Delta_i = A + B_i$, for some $\mathbb{Q}$-divisors $B_i \geq 0$. Assume that $D_i = K_X + \Delta_i$ is divisorially log terminal and $\mathbb{Q}$-Cartier. Then the Cox ring

$$R(\pi, D^\bullet) = \bigoplus_{m \in \mathbb{N}^k} \pi_* \mathcal{O}_X(\sum m_i D_{i,i}),$$

is a finitely generated $\mathcal{O}_U$-module.

1.2. Moduli Spaces. The main result under this heading concerns the existence of moduli spaces for varieties of general type. The quintessential results in this direction are Grothendieck’s proof [10] of the existence of a moduli space for smooth curves of genus $g > 1$, and the existence of a geometrically meaningful compactification by stable curves, due to Deligne and Mumford [4]. Some of these ideas and results were generalised to higher dimensions in a series of papers by Alexeev, Hassett, Karu, Kollár, Kovács, Shepherd-Barron, Viehweg and others.

We need some definitions:

Definition 1.2.1. $(X, \Delta)$ is called a pair if $X$ is $S_2$ and $K_X + \Delta$ is $\mathbb{Q}$-Cartier, where the coefficients of $\Delta$ lie between zero and one.

Fix a hilbert polynomial $h$ and a positive integer $r$. We say that a pair $(X, \Delta)$ has type $(h, r)$ if

1. $K_X + \Delta$ is semi-log canonical,
2. $r(K_X + \Delta)$ is Cartier,
3. $h(t) = \chi(\mathcal{O}_X(rt(K_X + \Delta)))$, and
4. $K_X + \Delta$ is ample.

We say that a flat morphism $\pi: Y \rightarrow Z$ of schemes, where $(Y, \Gamma)$ is a pair, is a family of log pairs of type $(h, r)$ if every reduced fibre $(Y_t, \Gamma_t)$ has type $(h, r)$. We say that
(6) \((X, \Delta)\) is smoothable of index \(r\), if there is a flat family over a smooth curve \(Z\), such that one reduced fibre \((Y_0, \Gamma_0)\) is isomorphic to \((X, \Delta)\) and the general reduced fibre \((Y_t, \Gamma_t)\) is kawamata log terminal and \(r(K_{Y_t} + \Gamma_t)\) is Cartier.

We say that a flat morphism \(\pi: Y \to Z\) of schemes, where \((Y, \Gamma)\) is a pair, is a family of pairs of type \((h, r)\) if \(\pi\) is a family of pairs of type \((h, r)\) (so that every component of \(\Gamma\) dominates \(Z\)) and the reduced fibres are smoothable of index \(r\).

We say that a reduced algebraic space \(M\) is a coarse moduli space, in the category of varieties, for smoothable pairs of type \((h, r)\), if there is a positive integer \(m\) such that

(i) For every algebraically closed field \(k\) of characteristic zero, the set of all \(k\)-points of \(M\) is equal to the set of isomorphism classes of polarised pairs

\[
\{(X, \mathcal{O}_X(m(K_X + \Delta))) \mid \text{where } (X, \Delta) \text{ is a smoothable pair of type } (h, r) \text{ over } k\},
\]

and

(ii) given any flat family of smoothable pairs \(\pi: Y \to Z\), where \(Z\) is a variety, then \(\mathcal{O}_Y(m(K_Y + \Gamma))\) is a line bundle and there is a unique morphism \(Z \to M\) for which the image of the closed point \(z \in Z\) is the isomorphism class of the fibre.

Moreover \(M\) is universal amongst all such objects satisfying (i) and (ii).

We can show the existence of a coarse moduli space for varieties of log general type:

**Corollary 1.2.2.** Fix a positive integer \(r\) and a hilbert function \(h\).

Then there is a positive integer \(m\) and a reduced algebraic space \(M = M_h^{sm}\), which is a coarse moduli space, in the category of varieties, for smoothable varieties \(X\) of type \((h, r)\). Every proper subspace of \(M\) is a projective variety. Further there is a surjective morphism \(U \to M\), where \(U\) is a projective variety. In particular, \(M\) is a projective variety in the absolute case (that is when \(\Delta\) is empty).

Moreover for each \(k\) sufficiently large and divisible there is a line bundle \(L_k\) on \(M_h^{sm}\) with the following property. Given any family \(\pi: Y \to Z\) of type pairs \((h, r)\), there is a finite morphism \(Z' \to Z\) such that for the induced morphism \(f: Z' \to M_h^{sm}\), \(f^*L_k\) is isomorphic to the determinant of the vector bundle \(\pi'^*\mathcal{O}_{Y'}(rk(K_{Y'} + \Gamma'))\), where \(\pi': Y' \to X'\) is the pullback of \(\pi\).

Note that we would like to prove a much stronger result than \(1.2.2\), we would like \(M_h^{sm}\) to be a coarse moduli space in the usual sense, that
is we would like to allow the base $B$ to be a scheme. In other words there ought to be a canonical scheme structure on $M^{sm}_h$. Unfortunately there does not seem to be any consensus on the correct definition of the moduli functor at present, and for this reason we have avoided this issue.

Note also that in general the moduli space $M$ is definitely not separated (when $\Delta$ is not empty), essentially because of the following:

**Corollary 1.2.3.** Fix a hilbert polynomial $h$.

Then there are positive integers $m$ and $p$, and a reduced algebraic space $P = P_{h}^{sm}$, which is a coarse moduli space, in the category of varieties, for polarised pairs $(X, L)$, such that

1. $L$ is ample,
2. $h(pt) = \chi(X, L^{\otimes t})$,
3. $L = \mathcal{O}_X(m(K_X + \Delta))$, where $(X, \Delta)$ is semi log canonical,
4. $(X, \Delta)$ is smoothable.

Every proper subspace of $P$ is a projective variety. Further there is a surjective morphism $U \to P$, where $U$ is a projective variety.

It would seem interesting to see if there are special cases when the moduli spaces constructed in (1.2.2) and (1.2.3) are separated.

At first sight (1.1.5) might seem a hard result to digest. For this reason, we would like to give a concrete, but non-trivial example. The moduli spaces $\overline{M}_{g,n}$ of $n$-pointed stable curves of genus $g$ are probably the most intensively studied moduli spaces. In particular the problem of trying to understand the related log canonical models via the theory of moduli has attracted a lot of attention (e.g. see [9], [30] and [15]).

**Corollary 1.2.4.** Let $X = \overline{M}_{g,n}$ the moduli space of stable curves of genus $g$ with $n$ marked points and let $\Delta_i$, $1 \leq i \leq k$ denote the boundary divisors.

Let $\Delta = \sum_i a_i \Delta_i$ be a boundary. Then $K_X + \Delta$ is log canonical and if $K_X + \Delta$ is big then there is a log canonical model $X \to Y$. Moreover if we fix a positive rational number $\delta$ and require that the coefficient $a_i$ of $\Delta_i$ is at least $\delta$ for each $i$ then the set of all log canonical models obtained this way is finite.

1.3. **Fano Varieties.** The next set of applications is to Fano varieties. The key observation is that given any divisor $D$, a small multiple of $D$ is linearly equivalent to a divisor of the form $K_X + \Delta$, where $\Delta$ is big and $K_X + \Delta$ is kawamata log terminal.

Using this observation we can show:
Corollary 1.3.1. Let $\pi : X \rightarrow U$ be a projective morphism of normal varieties, where $U$ is affine. Suppose that $X$ is $\mathbb{Q}$-factorial, $K_X + \Delta$ is divisorially log terminal and $-(K_X + \Delta)$ is ample over $U$.

Then $X$ is a Mori dream space.

There are many reasons why Mori dream spaces (see [16] for the definition) are interesting. As the name might suggest, they behave very well with respect to the Minimal Model Program. Given any divisor $D$, one can run the $D$-MMP, and this ends with either a nef model, or a fibration, for which $-D$ is relatively ample, and in fact any sequence of $D$-flips terminates.

(1.3.1) was conjectured in [16] where it is also shown that Mori dream spaces are GIT quotients of affine varieties by a torus. Moreover the decomposition given in (1.1.5) is induced by all the possible ways of taking GIT quotients, as one varies the linearisation.

Finally, it was shown in [16] that if one has a Mori dream space, then the Cox Ring is finitely generated.

We next prove a result that naturally complements (1.2). We show that if $K_X + \Delta$ is not pseudo-effective, then we can run the MMP with scaling to get a Mori fibre space:

Corollary 1.3.2. Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial kawamata log terminal pair. Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose that $K_X + \Delta$ is not $\pi$-pseudo-effective.

Then we may run $f : X \rightarrow Y$ a $(K_X + \Delta)$-MMP over $U$ that ends with a Mori fibre space $g : Y \rightarrow W$.

Finally we are able to prove a conjecture of Batyrev on the closed cone of nef curves for a Fano pair.

Definition 1.3.3. Let $X$ be a projective variety. A curve $\Sigma$ is called nef if $B \cdot \Sigma \geq 0$ for all effective Cartier divisors $B$. $\text{NF}(X)$ denotes the cone of nef curves sitting inside $H_2(X, \mathbb{R})$ and $\overline{\text{NF}}(X)$ denotes its closure.

Now suppose that $(X, \Delta)$ is a log pair. A $(K_X + \Delta)$-co-extremal ray is an extremal ray $F$ of the closed cone of nef curves $\overline{\text{NF}}(X)$ on which $K_X + \Delta$ is negative.

Corollary 1.3.4. Let $(X, \Delta)$ be a projective $\mathbb{Q}$-factorial kawamata log terminal pair such that $-(K_X + \Delta)$ is ample.

Then $\overline{\text{NF}}(X)$ is a rational polyhedron. If $F = F_i$ is a co-extremal ray of the closed cone of nef curves then there exists a divisor $\Theta$ such that the pair $(X, \Theta)$ is kawamata log terminal and we may run a $(K_X + \Theta)$-MMP $\pi : X \rightarrow Y$ which ends with a Mori fibre space $f : Y \rightarrow Z$. 

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such that $F$ is spanned by the pullback to $X$ of the class of any curve $\Sigma$ which is contracted by $f$.

1.4. Birational Geometry. Another immediate consequence of (1.2) is the existence of flips:

**Corollary 1.4.1.** Let $(X, \Delta)$ be a kawamata log terminal pair and let $\pi: X \to Z$ be a small $(K_X + \Delta)$-extremal contraction.

Then the flip of $\pi$ exists.

As already noted, we are unable to prove the termination of flips in general. However, using (1.1.5), we can show that any sequence of flips for the MMP with scaling terminates:

**Corollary 1.4.2.** Let $\pi: X \to U$ be a projective morphism of normal quasi-projective varieties. Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial kawamata log terminal pair, where $K_X + \Delta$ is $\mathbb{R}$-Cartier. Suppose that $\Delta$ is $\pi$-big and $K_X + \Delta$ is $\pi$-pseudo-effective.

If $K_X + \Delta + tC$ is kawamata log terminal and $\pi$-nef, then we may run the $(K_X + \Delta)$-MMP over $U$ with scaling of $C$.

Another application of (1.2) is the following result concerning the factorisation of certain birational maps:

**Corollary 1.4.3.** Let $(X, \Delta)$ be a log canonical pair and let $f: Y \to X$ be a log resolution. Suppose that there is a divisor $\Delta_0$ such that $K_X + \Delta$ is kawamata log terminal. Let $E$ be any set of exceptional divisors which satisfies the following two properties:

1. $E$ contains only exceptional divisors of log discrepancy at most one, and
2. the image of every exceptional divisor of log discrepancy one in $E$ does not contain any log canonical centre.

Then we may find a birational factorisation $\pi: W \to X$ of $f$, such that $\pi$ extracts precisely those exceptional divisors contained in $E$.

For example, if we assume that $(X, \Delta)$ is kawamata log terminal and we let $E$ be the set of all exceptional divisors with log discrepancy $\leq 1$, then the factorisation $\pi: W \to X$ of $f$ defined in (1.4.3) above is a terminal model of $(X, \Delta)$. In particular there is an effective $\mathbb{R}$ divisor $\Psi$ on $W$ such that $K_W + \Psi = \pi^*(K_X + \Delta)$ and $(W, \Psi)$ is a terminal pair.

If instead we assume that $(X, \Delta)$ is kawamata log terminal but we let $E$ be empty, then the factorisation $\pi: W \to X$ of $f$ defined in (1.4.3) above is a log terminal model. In particular $\pi$ is small, but $W$ is $\mathbb{Q}$-factorial and there is an effective $\mathbb{R}$ divisor $\Psi$ on $W$ such that $K_W + \Psi = \pi^*(K_X + \Delta)$.  

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However the result in (1.4.3) is not optimal as it does not fully address the log canonical case. Nevertheless, we are able to prove the following result (cf. [37], [28], [18]):

**Corollary 1.4.4 (Inversion of Adjunction).** Let $(X, \Delta)$ be a log pair and let $S$ be the normalisation of a component of $\Delta$ of coefficient one. Let $\Theta$ be the different.

Then the log discrepancy of $K_S + \Theta$ is equal to the log discrepancy of $K_X + \Delta$ in a neighbourhood of $S$.

One of the most compelling reasons to enlarge the category of varieties to the category of algebraic spaces (equivalently Moishezon spaces, at least in the proper case) is to allow the possibility of cut and paste operations, such as one can perform in topology. Unfortunately, it is then all too easy to construct proper smooth algebraic spaces over $\mathbb{C}$, which are not projective. In fact the appendix to [14] has two very well known examples due to Hironaka. In both examples, one exploits the fact that for two curves in a threefold which intersect in a node, the order in which one blows up the curves is important (in fact the resulting threefolds are connected by a flop).

It is then natural to wonder if this is the only way to construct such examples, in the sense that if a proper algebraic space is not projective then it must contain a rational curve. Kollár dealt with the case when $X$ is a terminal threefold with Picard number one, see [26]. In a slightly different but related direction, it is conjectured that if a complex Kähler manifold $M$ does not contain any rational curves then $K_M$ is nef (see for example [30]), which would extend some of Mori’s famous results from the projective case. Kollár also has some unpublished proofs of these results.

The following result, which was proved by Shokurov assuming the existence and termination of flips, cf. [39], gives an affirmative answer to the first conjecture and at the same time connects the two conjectures:

**Corollary 1.4.5.** Let $\pi : X \to U$ be a proper map of normal algebraic spaces, where $X$ is analytically $\mathbb{Q}$-factorial.

If $K_X + \Delta$ is divisorially log terminal and $\pi$ does not contract any rational curves then $\pi$ is a log terminal model. In particular $\pi$ is projective and $K_X + \Delta$ is $\pi$-nef.

2. Description of the Proof

**Theorem A (Existence of pl-flips).** Let $f : X \to Z$ be a pl-flipping contraction for an $n$-dimensional purely log terminal pair $(X, \Delta)$.

Then the flip $f^+ : X^+ \to Z$ of $f$ exists.
**Theorem B** (Existence of log terminal models). Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties, where $X$ has dimension $n$. Suppose that $K_X + \Delta$ is kawamata log terminal, where $\Delta$ is big over $U$.

If there exists an $\mathbb{R}$-divisor $D$ such that $K_X + \Delta \sim_{\mathbb{R},U} D \geq 0$, then $K_X + \Delta$ has a log terminal model over $U$.

**Theorem C** (Finiteness of models). Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties, where $X$ has dimension $n$. Fix $A$, a general ample $\mathbb{Q}$-divisor over $U$. Suppose that $K_X + \Delta_0$ is kawamata log terminal, for some $\Delta_0$.

Let $C \subset \mathcal{L}_A$ be a rational polytope such that either

1. For every $\Delta \in C$, $K_X + \Delta$ is $\pi$-big, or
2. $C = \mathcal{L}_A$.

Then the set of isomorphism classes

$$\{ Y \mid Y \text{ is the weak log canonical model over } U \text{ of a pair } (X,\Delta), \text{ where } \Delta \in C \}$$

is finite.

**Theorem D** (Non-vanishing theorem). Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties, where $X$ has dimension $n$. Suppose that $K_X + \Delta$ is kawamata log terminal, where $\Delta$ is big over $U$.

If $K_X + \Delta$ is $\pi$-pseudo-effective, then there exists an $\mathbb{R}$-divisor $D$ such that $K_X + \Delta \sim_{\mathbb{R},U} D \geq 0$.

The proof of Theorem A, Theorem B, Theorem C and Theorem D proceeds by induction:

- Theorem B$_{n-1}$, Theorem C$_{n-1}$ and Theorem D$_{n-1}$ imply Theorem A$_n$, see the main result of [12].
- Theorem B$_{n-1}$, Theorem C$_{n-1}$ and Theorem A$_n$ imply Theorem B$_n$, cf. (5.4).
- Theorem B$_n$ implies Theorem C$_n$, with condition (1), cf. (6.3).
- Theorem C$_{n-1}$, Theorem D$_{n-1}$, Theorem B$_n$ and Theorem C$_n$, with condition (1), imply Theorem D$_n$, cf. (7.4).
- Theorem B$_n$ and Theorem D$_n$ imply Theorem C$_n$, cf. (6.3).

2.1. **Sketch of the proof.** To help the reader navigate through the technical problems which arise naturally when trying to prove (1.2), we review a natural approach to proving that the canonical ring

$$R(X,K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X,\mathcal{O}_X(mK_X)),$$
of a smooth projective variety $X$ of general type is finitely generated. Even though we do not directly follow this method to prove the existence of log terminal models, instead using ideas from the MMP, many of the difficulties which arise in our approach are mirrored in trying to prove finite generation directly.

A very natural way to proceed is to pick a divisor $B \in |kK_X|$, whose existence is guaranteed as we are assuming that $K_X$ is big, and then to restrict to $B$. One obtains an exact sequence

$$0 \rightarrow H^0(X, \mathcal{O}_X((m(1+k)-k)K_X)) \rightarrow H^0(X, \mathcal{O}_X(m(1+k)K_X)) \rightarrow H^0(B, \mathcal{O}_B(mK_B)),$$

and it is easy to see that it suffices to prove that the restricted algebra, given by the image of the maps

$$H^0(X, \mathcal{O}_X(m(1+k)K_X)) \rightarrow H^0(B, \mathcal{O}_B(mK_B)),$$

is finitely generated. Various problems arise at this point. First $B$ is neither smooth nor even reduced (which, for example, means that the symbol $K_B$ is only formally defined; strictly speaking we ought to work with the dualising sheaf $\omega_B$). It is natural then to pass to a log resolution, so that the support $D$ of $B$ has simple normal crossings, and to replace $B$ by $D$. The second problem is that the kernel of the map

$$H^0(X, \mathcal{O}_X(m(1+k)K_X)) \rightarrow H^0(D, \mathcal{O}_D(mK_D)),$$

no longer has any obvious connection with $H^0(X, \mathcal{O}_X((m(1+k)-k)K_X))$, so that even if we knew that the new restricted algebra were finitely generated, it is not immediate that this is enough. Another significant problem is to identify the restricted algebra as a subalgebra of $\bigoplus_{m \in \mathbb{N}} H^0(D, \mathcal{O}_D(mK_D))$, since it is only the latter that we can handle by induction. Yet another problem is that if $C$ is a component of $D$, it is no longer the case that $C$ is of general type, so that we need a more general induction. In this case the most significant problem to deal with is that even if $K_C$ is pseudo-effective, it is not clear that the linear system $|kK_C|$ is non-empty for any $k > 0$. Finally, even though this aspect of the problem may not be apparent from the description above, in practice it seems as though we need to work with infinitely many different values of $k$ and hence $B = B_k$, which entails working with infinitely many different birational models of $X$ (since for every different value of $k$, one needs to resolve the singularities of $D$).

We now review our approach to the proof of (1.2). As is clear from the plan of the proof given in the previous subsection, the proof of (1.2) is by induction on the dimension and the proof is split into four parts. However instead of proving that the canonical ring is finitely generated, we try to construct a minimal model for $X$. The first part is to prove
the existence of pl-flips. This is proved by induction in [11], and we will not talk about the proof of this result here, since the methods used to prove this result are very different from the methods we use here. Granted existence of pl-flips, the main issue is to prove that the MMP terminates, which means that we must show that we only need finitely many flips.

As in the scheme of the proof of finite generation sketched above, the first step is to pick $B \in |K_X|$, and to pass to a log resolution of the support $D$ of $B$. By way of induction we want to work with $K_X + D$ rather than $K_X$. As before this is tricky since a minimal model for $K_X + D$ is not the same as a minimal model for $K_X$. In other words having added $D$, we really want to subtract it as well. The trick however is to first add $D$, construct a minimal model for $K_X + D$ and then subtract $D$ (almost literally component by component). This is the key step, to show that Theorem B, Theorem C and Theorem A imply Theorem B. This part of the proof splits naturally into two parts. First we have to prove that we may run the relevant minimal model programs, see §4 and then we have to prove this does indeed construct a minimal model for $K_X$, see §5.

To gain intuition for how this part of the proof works, let us consider a series of successively harder cases. First suppose that $D = S$ is irreducible. In this case it is clear that $S$ is of general type and $K_X$ is nef iff $K_X + S$ is nef and in fact a log terminal model for $K_X$ is the same as a log terminal model for $K_X + S$. Consider running the $(K_X + S)$-MMP. Then every step of this MMP is a step of the $K_X$-MMP and vice-versa. Suppose that we have a $(K_X + S)$-extremal ray $R$. Let $\pi: X \longrightarrow Z$ be the corresponding contraction. Then $S \cdot R < 0$, so that every curve $\Sigma$ contracted by $\pi$ must be contained in $S$. If $\pi$ is a divisorial contraction, then $\pi$ must contract $S$ and $K_Z \sim Q 0$, so that $Z$ is a log terminal model. Otherwise $\pi$ is a pl-flip and by Theorem A, we can construct the flip of $\pi$, $\phi: X \longrightarrow Y$. Consider the restriction $\psi: S \longrightarrow T$ of $\phi$ to $S$, where $T$ is the strict transform of $S$. Since log discrepancies increase under flips and $S$ is irreducible, $\psi^{-1}$ cannot contract any divisors. After finitely many flips, we may therefore assume that $\psi$ does not contract any divisors, since the Picard number of $S$ cannot keep dropping. Consider what happens if we restrict to $S$. By adjunction, we have

$$(K_X + S)|_S = K_S.$$  

Thus $\psi: S \longrightarrow T$ is $K_S$-negative. We have to show that this cannot happen infinitely often. If we knew that every sequence of flips on $S$ terminates, then we would be done. In fact this is how special termination works. Unfortunately we cannot prove that every sequence of flips
terminates on $S$, so that we have to do something slightly different. Instead we throw in an auxiliary ample divisor $H$ on $X$, and consider $K_X + S + tH$, where $t$ is a positive real number. If $t$ is large enough then $K_X + S + tH$ is ample. Decreasing $t$, we may assume that there is an extremal ray $R$ such that $(K_X + S + tH) \cdot R = 0$. If $t = 0$, then $K_X + S$ is nef and we are done. Otherwise $(K_X + S) \cdot R < 0$, so that we are still running a $(K_X + S)$-MMP, but with the additional restriction that $K_X + S + tH$ is nef and trivial on any ray we contract. Let $G = H|_S$. Then $K_S + tG$ is nef and so is $K_T + tG'$, where $G' = \psi_* G$. In this case $K_T + tG'$ is a weak log canonical model for $K_S + tG$ (it is not a log terminal model, both because $\phi$ might contract divisors on which $K_S + tG$ is trivial and more importantly because $T$ need not be $\mathbb{Q}$-factorial). In this case we are then done, by finiteness of weak log canonical models for $(S, tG)$, where $t \in [0, 1]$ (cf. Theorem C in -1).

Now we consider a slightly more general case, where $K_X \sim_{\mathbb{Q}} sS = s \sum S_i$, for some rational number $s$. The main problem here is that not every component $S_i$ of $S$ is of general type, so the induction on dimension is a little more involved. On the other hand, since $K_X$ is big, we can always pick $\Delta \sim_{\mathbb{Q}} \epsilon K_X$, for some $\epsilon > 0$, such that $\Delta = A + B$, where $A$ is ample and $B \geq 0$ and $K_X + \Delta$ is kawamata log terminal. Note that a log terminal model for $K_X$ is the same as a log terminal model for $K_X + A + B$. Assuming that no component of $\Delta$ is a component of $S$, a log terminal model for $K_X + A + B$ is the same as a log terminal model for $K_X + S + A + B$. As before if $R$ is an extremal ray which is $(K_X + S + A + B)$-negative, then $S \cdot R < 0$, so that $S_i \cdot R < 0$ for some component $S_i$ of $S$. As before this guarantees existence of flips. Now if the $(K_X + S + A + B)$-MMP does not terminate, then we must have an infinite sequence of flips which must intersect one component $S_i$ of $S$ infinitely often. Using the ample divisor $A$ to tie break, and since the only issue is termination, we may assume that $S$ is irreducible.

The first added complication is that if $\psi: S \to T$ is the restriction of a flip $\phi: X \to Y$ to $S$, then $\psi^{-1}$ might contract some divisors. However it is a standard result that this can only happen finitely many times. As before, we run a $(K_X + S + A + B)$-MMP, with scaling of $H$, so that we only consider $(K_X + S + A + B)$-extremal rays $R$, which are $(K_X + S + A + B + tH)$-trivial, where $K_X + S + A + B + tH$ is nef. If we restrict to $S$, then we have

$$(K_X + S + A + B)|_S = K_S + C + D,$$

where $C = A|_S$ is ample and $D = B|_S \geq 0$. As $K_X + S + A + B$ is purely log terminal, $K_S + C + D$ is kawamata log terminal, so that we
have exactly the right hypotheses to apply induction, and we can use finiteness of weak log canonical models for $S$ to finish, as before.

Now consider an even more general case, $K_X \sim \sum s_i S_i$, where $s_1, s_2, \ldots, s_k$ are not necessarily equal. Note that if $S = \sum S_i$, then $K_X + S + A + B \sim D \geq 0$, where the support of $D$ is equal to $S$. Thus if $(K_X + S + A + B) \cdot R < 0$, then $D \cdot R < 0$, so that $S_i \cdot R < 0$, for some component of $S_i$ of $S$. Thus, by the same argument as before, we may construct a log terminal model for $K_X + S + A + B$. The main new difficulty is that a log terminal model for $K_X + S + A + B$ is not necessarily a log terminal model for $K_X$. However it is not too hard to get around this problem. For $k$ sufficiently large, we may decompose $D$ as $M + F$, where $M$ corresponds to the mobile part of $|kK_X|$ for some $k$ (so that $M$ is either irreducible or empty) and the components of $F$ are all components of the stable base locus. Then replace $X$ by a log resolution, so that the support of $S + A + B$ has global normal crossings. Running the $(K_X + S + A + B)$-MMP with scaling of $H$, as before, we may assume that $K_X + S + A + B$ is nef. At this stage we want to subtract $M$. So now we run the $(K_X + F + A + B)$-MMP, with scaling of $M$. Suppose that $R$ is a $(K_X + F + A + B)$-negative ray. By definition of the MMP with scaling, $M \cdot R > 0$. But then $F_i \cdot R < 0$ for some component $F_i$ of $F$. At the end, we have that $K_X + F + A + B$ is nef. Since $A$ is big, by the base point free theorem, $K_X + F + A + B$ is therefore semiample. Since every component of $F$ is part of the stable base locus of $K_X + A + B$, it is also part of the stable base locus of $K_X + F + A + B$, and it must be the case that $F$ is empty, that is, it has been contracted by the two MMPs we have just run. Finally, we need to observe that since we only contracted divisors on which $K_X + A + B + F$ is negative, even though we did not run a $K_X + A + B$-MMP, in fact we still constructed a log terminal model for $K_X + A + B$, which is the same a log terminal model for $K_X$.

Now we consider the most general case. In this case we need to start with $K_X + \Delta$, where $\Delta = A + B$ is an $\mathbb{R}$-divisor (we will see later why it is necessary to work at this level of generality). In this case $K_X + \Delta \sim \sum s_i S_i$. Assume that $B$ and $S$ have no common components (we only assume this to simplify the notation). As before the first step is to run a MMP so that $K_X + S + A + B$ is nef. The argument is almost exactly the same as before, except that if we try to write $D = M + F$, since we are now dealing with $\mathbb{R}$-divisors, we can no longer assume that $M$ is irreducible and we need to work up to linear equivalence. The best we can arrange for is that every component of $M$ is mobile. With this choice of $M$, at least it is the case that no component can ever be contracted, by running a MMP. So now the
process of subtracting $M$ is much more delicate. In fact we need to subtract each component of $M$ one at a time.

To get a feeling for how this proceeds consider two special cases. To warm up, suppose first that $M = m_1 M_1 + m_2 M_2$ has two components. Let $N = n_1 M_1 + n_2 M_2$ be the part of $K_X + S + A + B$ corresponding to the support of $M$, so that to begin with we have $n_1 = n_2 = 1$. Possibly switching $M_1$ and $M_2$ we may assume that $m_1 < m_2$ (if $m_1 = m_2$ there is nothing to do). By subtracting part of $M_1$, we want to end up with $N = \lambda M$. In this case $\lambda = 1/m_2$. We have

$$K_X + (S - C) + A + B + C = D + C,$$

is nef, where $C = M_1$ and $D \geq 0$. The key point is that every $K_X + (S - C) + A + B$-negative contraction is automatically negative with respect to $S - C$, so that we may run the $K_X + (S - C) + A + B$-MMP with scaling of $C$ until the coefficient of $C = M_1$ is $m_1/m_2$. At this point

$$N = 1/m_2 M_1 + M_2 = 1/m_2(m_1 M_1 + m_2 M_2),$$

as desired. Then we can finish off as before. Now suppose that there are three components, $M = m_1 M_1 + m_2 M + m_3 M_3$. We may assume that $m_1 < m_2 < m_3$. We have $N = n_1 M_1 + n_2 M_2 + n_3 M_3$, we start with $n_1 = n_2 = n_3 = 1$ and we would like to end with $N = \lambda M$ by subtracting components of $S$. In this case $\lambda = 1/m_3$, so that we know that we would like to end with the coefficients equal to $n_1 = m_1/m_3$, $n_2 = m_2/m_3$ and $n_3 = 1$. The tricky thing is that we only know existence and termination of flips for contractions whose flipping curve is contained in the reduced boundary. Suppose that we are able to write

$$K_X + S + A + B + C \sim_R D + \alpha C,$$

where $\alpha \geq 0$, $D \geq 0$ and the support of $D$ is contained in $S$. Then when we run the $K_X + S + A + B$-MMP with scaling of $C$, $C$ will be positive on every ray, so that $D$ is negative on every ray and so every flipping curve is automatically contained in the boundary.

The first step will be to run a MMP with scaling of $C = M_1$ (so that $\alpha = m_1$) and the last step will be to run a MMP with scaling of $c_1 M_1 + c_2 M_2$, for a choice of $c_1$ and $c_2$ which is to be determined. Suppose that we stop the first step when the coefficient of $M_1$ is $t$. At this point we may write

$$K_X + \Delta \sim_R D - (1 - t) M_1.$$

If we want to write this in the form

$$K_X + S + A + B + C \sim_R D_1 + \alpha C,$$
where the support of $D_1$ is contained in $S$, $\alpha \geq 0$ and the coefficient of $M_1$ and $M_2$ in $C$ is $c_1$ and $c_2$, then we must take

$$c_1 = t \quad \text{and} \quad c_2 = 1.$$ 

After rescaling $C$ to $sC$, we want

$$st = sc_1 = m_1/m_3 \quad \text{and} \quad s = m_2/m_3.$$ 

Using these equations we see that we must stop the first stage when $t = m_1/m_2$. The rest of the argument proceeds as before. In general, when $M$ has more components, we must perform a similar, but more involved analysis. In particular, to calculate when to stop any particular stage, we need to determine what happens at all of the preceding stages. The details are contained in §5.

Now we explain the rest of the proof. In terms of induction, we need to prove finiteness of weak log canonical models. We fix an ample divisor $A$ and work with divisors of the form $K_X + \Delta = K_X + A + B$, where the coefficients of $B$ are variable. For ease of exposition, we assume that the support of $A$ and $B$ have global normal crossings, so that $K_X + \Delta = K_X + A + \sum b_iB_i$ is log canonical iff $0 \leq b_i \leq 1$ for all $i$. The key point is that we allow the coefficients of $B$ to be real numbers, so that the set of all possible choices of coefficients $[0, 1]^k$ is a compact subset of $\mathbb{R}^k$. Thus we may check finiteness locally. In fact since $A$ is ample, we can always perturb the coefficients of $B$ so that none of the coefficients are equal to one or zero and so we may even assume that $K_X + \Delta$ is kawamata log terminal.

Observe that we are certainly free to add components to $B$ (formally we add components with coefficient zero and then perturb so that their coefficients are non-zero). In particular we may assume that $B$ is the support of an ample divisor and so working on the weak log canonical model, we may assume that we have a log canonical model for a perturbed divisor. Thus it suffices to prove that there are only finitely many log canonical models. Since the log canonical model is determined by any log terminal model, it suffices to prove that we can find a cover of $[0, 1]^k$ by finitely many log terminal models. By compactness, it suffices to do this locally.

So pick $b \in [0, 1]^k$. There are two cases. If $K_X + \Delta$ is not pseudo-effective, then $K_X + A + B'$ is not pseudo-effective, for $B'$ in a neighbourhood of $B$, and there are no weak log canonical models at all. Otherwise we may assume that $K_X + \Delta$ is pseudo-effective. Suppose that we know that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$. Then we know that there is a log terminal model $\phi: X \dashrightarrow Y$. Replacing $(X, \Delta)$ by $(Y, \Gamma = \phi_*\Delta)$, we may assume that $K_X + \Delta$ is nef. By the base point free theorem it
is semiample. Let $X \to Z$ be the corresponding morphism. The key observation is that locally about $\Delta$, any log terminal model over $Z$ is an absolute log terminal model. Working over $Z$, we may assume that $K_X + \Delta$ is numerically trivial. In this case the problem of finding a log terminal model for $K_X + \Delta'$ only depends on the line segment spanned by $\Delta$ and $\Delta'$. Working in a small box about $\Delta$, we are then reduced to finding a log terminal model on the boundary of the box and we are done by induction on the dimension of the affine space containing $B$. Note that in practice, we need to work in slightly more generality than we have indicated; first we need to work in the relative setting and secondly we need to work with an arbitrary affine space containing $B$ (and not just the space spanned by the components of $B$). This poses no significant problem. This explains the implication Theorem $\text{C}_{n-1}$ and Theorem $\text{B}_n$ imply Theorem $\text{C}_n$. The details are contained in §6.

Finally we need to explain how to prove that if $K_X + \Delta = K_X + A + B$ is pseudo-effective, then $K_X + \Delta \sim_R D \geq 0$. The idea is to mimic the proof of the non-vanishing theorem. As in the proof of the non-vanishing theorem and following the work of Nakayama, there are two cases. In the first case, for any ample divisor $H$,

$$h^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor + H)),$$

is a bounded function of $m$. It is not hard to prove that Theorem $\text{B}_n$ implies that $K_X + \Delta$ has a log terminal model and we are done by the base point free theorem.

In the second case we construct a log canonical centre for $m(K_X + \Delta) + H$, when $m$ is sufficiently large. Passing to a log resolution, and using standard arguments, we are reduced to the case when $K_X + \Delta = K_X + S + A + B$, where $S$ is irreducible and $(K_X + \Delta)|_S$ is pseudo-effective, and the support of $\Delta$ has global normal crossings. Suppose first that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We may write

$$(K_X + S + A + B)|_S = K_S + C + D,$$

where $C$ is ample and $D \geq 0$. By induction we know that there is a positive integer $m$ such that $h^0(S, \mathcal{O}_S(m(K_S + C + D))) > 0$. To lift sections, we need to know that $h^1(X, \mathcal{O}_X(m(K_X + S + A + B) - S)) = 0$. Now

$$m(K_X + S + A + B) - (K_X + B) - S = (m - 1)(K_X + S + A + B) + A$$

$$= (m - 1)(K_X + \Delta + \frac{1}{m-1}A).$$
As $K_X + \Delta + A/(m - 1)$ is big, we can construct a log terminal model $\phi: X \rightarrow Y$ for $K_X + \Delta + A/(m - 1)$, and running this argument on $Y$, the required vanishing holds by Kawamata-Viehweg vanishing. In the general case, $K_X + S + A + B$ is an $R$-divisor. The argument is now a little more delicate as $h^0(S, O_S(m(K_S + C + D)))$ does not make sense. We need to approximate $K_S + C + D$ by rational divisors, which we can do by induction. But then it is not so clear how to choose $m$. In practice we need to prove that the log terminal model $Y$ constructed above does not depend on $m$, at least locally in a neighbourhood of $T$, the strict transform of $S$, and then the result follows by Diophantine approximation. This explains the implication Theorem $[\text{C}_n-1]$, Theorem $[\text{B}]$, and Theorem $[\text{C}]$ with condition (1) imply Theorem $[\text{D}]$. The details are in [§7].

2.2. Standard conjectures of the MMP. Having sketched the proof of (1.2), we should point out the main obstruction to extending these ideas to the case when $X$ is not of general type. The main issue seems to be the implication $K_X$ pseudo-effective implies $\kappa(X, K_X) \geq 0$. In other words we need:

**Conjecture 2.1.** Let $(X, \Delta)$ be a projective kawamata log terminal pair. 
If $K_X + \Delta$ is pseudo-effective then $\kappa(X, K_X + \Delta) \geq 0$.

We also probably need

**Conjecture 2.2.** Let $(X, \Delta)$ be a projective kawamata log terminal pair. 
If $K_X + \Delta$ is pseudo-effective and 

$$h^0(X, \mathcal{O}_X(\lceil m(K_X + \Delta) \rceil + H)),$$

is not a bounded function of $m$, for some ample divisor $H$, then $\kappa(X, K_X + \Delta) \geq 1$.

In fact, using the methods of this paper, together with some results of Kawamata (cf. [20] and [21]), (2.1) and (2.2) would seem to imply one of the main outstanding conjectures of higher dimensional geometry:

**Conjecture 2.3** (Abundance). Let $(X, \Delta)$ be a projective kawamata log terminal pair. 
If $K_X + \Delta$ is nef then it is semiample.

We remark that the following seemingly innocuous generalization of (1.2) (in dimension $n$) would seem to imply (2.3) (in dimension $n + 1$).
Conjecture 2.4. Let \((X, \Delta)\) be a projective log canonical pair of dimension \(n\).
If \(K_X + \Delta\) is big, then \((X, \Delta)\) has a log canonical model.

It also seems worth pointing out that the other remaining conjecture is:

Conjecture 2.5 (Borisov-Alexeev-Borisov). Fix a positive integer \(n\) and a positive real number \(\epsilon > 0\).
Then the set of varieties \(X\) such that \(K_X + \Delta\) has log discrepancy at least \(\epsilon\) and \(-(K_X + \Delta)\) is ample, forms a bounded family.

3. Preliminary Results

In this section we collect together some definitions and results that will be needed for the proof of 1.4.2.

3.1. Notation and conventions. We work over the field of complex numbers \(\mathbb{C}\). We say that two \(\mathbb{Q}\)-divisors \(D_1, D_2\) are \(\mathbb{Q}\)-linearly equivalent \((D_1 \sim_\mathbb{Q} D_2)\) if there exists an integer \(m > 0\) such that \(mD_i\) are linearly equivalent. We say that a \(\mathbb{Q}\)-divisor \(D\) is \(\mathbb{Q}\)-Cartier if some integral multiple is Cartier. We say that \(X\) is \(\mathbb{Q}\)-factorial if every Weil divisor is \(\mathbb{Q}\)-Cartier. We say that \(X\) is analytically \(\mathbb{Q}\)-factorial if every analytic Weil divisor (that is an analytic subset of codimension one) is analytically \(\mathbb{Q}\)-Cartier (i.e. some multiple is locally defined by a single analytic function). We recall some definitions involving divisors with real coefficients:

Definition 3.1.1. Let \(\pi: X \rightarrow U\) be a proper morphism of normal algebraic spaces.

1. An \(\mathbb{R}\)-Weil divisor \(D\) on \(X\) is an \(\mathbb{R}\)-linear combination of prime divisors.
2. An \(\mathbb{R}\)-Cartier divisor \(D\) is an \(\mathbb{R}\)-linear combination of Cartier divisors.
3. Two \(\mathbb{R}\)-divisors \(D\) and \(D'\) are \(\mathbb{R}\)-linearly equivalent over \(U\), denoted \(D \sim_{\mathbb{R}, U} D'\), if their difference is an \(\mathbb{R}\)-linear combination of principal divisors and an \(\mathbb{R}\)-Cartier divisor pulled back from \(U\).
4. An \(\mathbb{R}\)-Cartier divisor \(D\) is ample over \(U\) (or \(\pi\)-ample) if it is \(\mathbb{R}\)-linearly equivalent to a positive linear combination of ample (in the usual sense) divisors over \(U\).
5. A \(\mathbb{R}\)-Cartier divisor \(D\) on \(X\) is nef over \(U\) (or \(\pi\)-nef) if \(D \cdot C \geq 0\) for any curve \(C \subset X\), contracted by \(\pi\).
6. An \(\mathbb{R}\)-divisor \(D\) is effective if it is a positive real linear combination of prime divisors.
An \( \mathbb{R} \)-divisor \( D \) is \( \pi \)-effective if it is effective on the generic fibre.

An \( \mathbb{R} \)-divisor \( D \) is big over \( U \) (or \( \pi \)-big) if
\[
\limsup \frac{h^0(F, \mathcal{O}_F(\lfloor mD \rfloor))}{m^{\dim F}} > 0,
\]
for the fibre \( F \) over any generic point of \( U \).

An \( \mathbb{R} \)-Cartier divisor \( D \) is semiample over \( U \) (or \( \pi \)-semiample) if there is a morphism \( \pi: X \rightarrow Y \) over \( U \) such that \( D \) is \( \mathbb{R} \)-linearly equivalent to the pullback of an ample divisor over \( U \).

An \( \mathbb{R} \)-divisor \( D \) is pseudo-effective if the numerical class of \( D \) is a limit of effective divisors.

An \( \mathbb{R} \)-divisor \( D \) is \( \pi \)-pseudo-effective if the restriction of \( D \) to the generic fibre is pseudo-effective.

Note that the group of Weil divisors with rational or real coefficients forms a vector space, with a canonical basis given by the prime divisors. Given an \( \mathbb{R} \)-divisor, \( \|D\| \) denotes the sup norm with respect to this basis. If \( A \) and \( B \) are two \( \mathbb{R} \)-divisors, then we let \( [A,B] \) denote the line segment
\[
\{ \lambda A + \mu B \mid \lambda + \mu = 1, \lambda \geq 0, \mu > 0 \}.
\]

Given an \( \mathbb{R} \)-divisor \( D \) and a subvariety \( Z \) which is not contained in the singular locus of \( X \), \( \text{mult}_Z D \) denotes the multiplicity of \( D \) at the generic point of \( Z \). If \( Z = E \) is a prime divisor this is the coefficient of \( E \) in \( D \).

A log pair \( (X, \Delta) \) is a normal variety \( X \) and an effective \( \mathbb{R} \)-Weil divisor \( \Delta \) such that \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier. We say that a log pair \( (X, \Delta) \) is log smooth, if \( X \) is smooth and the support of \( \Delta \) is a divisor with global normal crossings. A projective morphism \( g: Y \rightarrow X \) is a log resolution of the pair \( (X, \Delta) \) if \( Y \) is smooth and \( g^{-1}(\Delta) \) union the exceptional set of \( g \) is a divisor with normal crossings support. We write
\[
K_Y + \Gamma = g^*(K_X + \Delta),
\]
and \( \Gamma = \sum a_i \Gamma_i \) where \( \Gamma_i \) are distinct reduced irreducible divisors. The log discrepancy of \( \Gamma_i \) is \( 1 - a_i \). The locus of log canonical singularities of the pair \( (X, \Delta) \), denoted LCS(\(X, \Delta)\), is equal to the image of those components of \( \Gamma \) of coefficient at least one (equivalently log discrepancy at most zero). The images of those components of \( \Gamma \) of coefficient at least one are the log canonical centers of the pair \( (X, \Delta) \). The pair \( (X, \Delta) \) is kawamata log terminal if for every (equivalently for one) log resolution \( g: Y \rightarrow X \) as above, the coefficients of \( \Gamma \) are strictly less than one, that is \( a_i < 1 \) for all \( i \). Equivalently, the pair \( (X, \Delta) \)
is kawamata log terminal if the locus of log canonical singularities is empty. We say that the pair \((X, \Delta)\) is purely log terminal if the log discrepancy of any exceptional divisor is greater than zero. We say that the pair \((X, \Delta)\) is divisorially log terminal if there is a log resolution such that the log discrepancy of every exceptional divisor is greater than zero.

We will also often write

\[ K_Y + \Gamma = g^*(K_X + \Delta) + E, \]

where \(\Gamma\) and \(E\) are effective, with no common components, \(g_*\Gamma = \Delta\) and \(E\) is \(g\)-exceptional. Note that this decomposition is unique.

**Definition 3.1.2.** Let \((X, \Delta)\) be a divisorially log terminal pair and \(f : X \to Z\) be a projective morphism of normal varieties. Then \(f\) is a pl-flipping contraction if

1. \(f\) is a small birational morphism of relative Picard number \(\rho(X/Z) = 1\),
2. \(-(K_X + \Delta)\) is \(f\)-ample,
3. \(X\) is \(\mathbb{Q}\)-factorial and \(\Delta\) is a \(\mathbb{R}\)-divisor,
4. \(\lfloor \Delta \rfloor = S + S'\) where \(S\) is irreducible and \(-S\) is \(f\)-ample.

The flip \(f^+ : X^+ \to Z\) of a (pl)-flipping contraction \(f : (X, \Delta) \to Z\) is a small birational projective morphism of normal varieties \(f^+ : X^+ \to Z\) such that \(K_{X^+} + ((f^+)^{-1} \circ f)_*\Delta\) is \(f^+\)-ample. Notice that \(X^+\) is the log canonical model of \((X, \Delta)\) over \(Z\) and that \(X^+\) is \(\mathbb{Q}\)-factorial and \(\rho(X^+/Z) = 1\).

### 3.2. Preliminaries.

**Lemma 3.2.1.** Let \(\pi : X \to U\) be a proper morphism of normal quasi-projective varieties. Let \(D\) be a \(\mathbb{R}\)-Cartier divisor on \(X\) and let \(D'\) be its restriction to the generic fibre of \(\pi\).

If \(D' \sim \mathbb{R} B' \geq 0\) for some \(\mathbb{R}\)-divisor \(B'\) on the generic fibre of \(\pi\), then there is a divisor \(B\) on \(X\) such that \(D \sim \mathbb{R}_U B \geq 0\).

**Proof.** Taking the closure of the generic points of \(B'\), we may assume that there is an effective \(\mathbb{R}\)-divisor \(B_1\) on \(X\) such that the restriction of \(B_1\) to the generic fibre is \(B'\). As

\[ D' - B' \sim \mathbb{R} 0, \]

it follows that there is an open subset \(U_1\) of \(U\), such that

\[ (D - B_1)|_{U_1} \sim \mathbb{R} 0, \]
where $V_1$ is the inverse image of $U_1$. But then there is a divisor $G$ on $X$ such that

$$D - B_1 \sim_{\mathbb{R}} G,$$

where $Z = \pi(\text{Supp} \, G)$ is a proper closed subset. As $U$ is quasi-projective, there is an ample divisor $H$ on $U$ which contains $Z$. Possibly rescaling, we may assume that $F = \pi^*H \geq -G$. But then

$$D \sim_{\mathbb{R}} (B_1 + F + G) - F,$$

so that

$$D \sim_{\mathbb{R}, U} (B_1 + F + G) \geq 0.$$

□

3.3. Nakayama-Zariski Decomposition. We will need some definitions and results from [34].

**Definition-Lemma 3.3.1.** Let $X$ be a smooth projective variety, $B$ be a big $\mathbb{R}$-divisor and let $C$ be a prime divisor. Let

$$\sigma_C(B) = \inf \{ \text{mult}_C(B') \mid B' \sim_{\mathbb{Q}} B, B' \geq 0 \}.$$

Then $\sigma_C$ is a continuous function on the cone of big divisors.

Now let $D$ be any pseudo-effective $\mathbb{R}$-divisor and let $A$ be any ample $\mathbb{Q}$-divisor. Let

$$\sigma_C(D) = \lim_{\epsilon \to 0} \sigma_C(D + \epsilon A).$$

Then $\sigma_C(D)$ exists and is independent of the choice of $A$.

There are only finitely many prime divisors $C$ such that $\sigma_C(D) > 0$. Set $N_\sigma(D) = \sum_C \sigma_C(D) C$, then $N_\sigma(D)$ is determined by $c_1(D)$.

**Proof.** The first statement is (3) of (2.1.1) of [34] and the rest is (2.1.6) of [34]. □

**Proposition 3.3.2.** Let $X$ be a smooth projective variety and let $D$ be a pseudo-effective $\mathbb{R}$-divisor. Let $B$ be any big $\mathbb{R}$-divisor.

If $D$ is not numerically equivalent to $N_\sigma(D)$, then there is a positive integer $k$ and a positive rational number $\beta$ such that

$$h^0(X, \mathcal{O}_X(\lfloor mD \rfloor + \lfloor kB \rfloor)) > \beta m,$$

for all $m \gg 0$.

**Proof.** Let $A$ be any integral divisor. Then we may find a positive integer $k$ such that

$$h^0(X, \mathcal{O}_X(\lfloor kB \rfloor - A)) \geq 0.$$ 

Thus it suffices to exhibit an ample divisor $A$ and a positive rational number $\beta$ such that

$$h^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A)) > \beta m \quad \text{for all} \quad m \gg 0.$$ 

Replacing $D$ by $D - N_\sigma(D)$, we may assume that $N_\sigma(D) = 0$. Now apply (6.1.12) of [34]. □
3.4. Log pairs. We recall some basic facts and standard manipulations for log pairs. First the following result, which is well known in the case of $\mathbb{Q}$-divisors and is due to Kawamata and Shokurov:

**Theorem 3.4.1** (Base Point Free Theorem). Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial kawamata log terminal pair, where $\Delta$ is an $\mathbb{R}$-divisor. Let $f : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties, and let $D$ be a nef $\mathbb{R}$-divisor over $U$, such that $aD - (K_X + \Delta)$ is nef and big over $U$, for some positive real number $a$.

Then $D$ is semiample over $U$.

**Proof.** As the property that $D$ is semiample over $U$ is local, we may assume that $U$ is affine. This case is (7.1) of [11] (for example). □

**Corollary 3.4.2.** Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial kawamata log terminal pair, where $\Delta$ is an $\mathbb{R}$-divisor. Let $f : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties such that $K_X + \Delta$ is nef over $U$ and $\Delta$ is big over $U$.

Then $K_X + \Delta$ is semiample over $U$.

**Proof.** As in the proof of (3.4.1), we may assume that $U$ is affine.

In this case, we may write $\Delta \sim_{\mathbb{R},U} A + B$, where $A$ is ample over $U$ and $B$ is effective. Let

$$\Delta' + \epsilon A = (1 - \epsilon)\Delta + \epsilon B + \epsilon A \sim_{\mathbb{R},U} \Delta.$$ 

Then for any $\epsilon > 0$ sufficiently small, $K_X + \Delta'$ is kawamata log terminal. On the other hand

$$(K_X + \Delta) - (K_X + \Delta') \sim_{\mathbb{R},U} \epsilon A,$$

and we may apply (3.4.1). □

We also recall some basic facts about adjunction and the different, see [28] for more details. Let $(X, \Delta)$ be a log canonical pair, and let $S$ be a normal component of $\Delta$ of coefficient one. Then there is a divisor $\Theta$ on $S$ such that

$$(K_X + \Delta)|_S = K_S + \Theta.$$ 

If $(X, \Delta)$ is divisorially log terminal then so is $K_S + \Theta$ and if $(X, \Delta)$ is purely log terminal then $K_S + \Theta$ is kawamata log terminal. The pair $(S, \Theta)$ only depends on a neighbourhood of $S$ in $X$.

**Lemma 3.4.3.** Let $\pi : X \rightarrow C$ be a flat projective morphism from a normal variety $X$ to the germ of a nonsingular curve $0 \in C$ such that $(X, \Delta)$ is a pair where $K_X + \Delta$ is $\mathbb{Q}$-Cartier. If the special fibre $(X_0, \Delta_0)$ is semi log canonical, then the general fibre $(X_t, \Delta_t)$ is log canonical.
Proof. It suffices to prove that $X$ is log canonical in a neighbourhood of $X_0$. This follows from [15] and the definition of semi log canonical. □

3.5. Types of model.

Definition 3.5.1. Let $\phi: X \dasharrow Y$ be a birational map, whose inverse does not contract any divisors, and let $D$ be a $\mathbb{R}$-Cartier divisor such that $D' = \phi_*D$ is also $\mathbb{R}$-Cartier. We say that $\phi$ is $D$-non-positive (respectively $D$-negative) if for some common resolution $p: W \to X$ and $q: W \to Y$, we may write

$$p^*D = q^*D' + E,$$

where $E \geq 0$ is $q$-exceptional (respectively $E \geq 0$ is $q$-exceptional and the support of $E$ contains the strict transform of the $\phi$-exceptional divisors).

We will often use the following well-known:

Lemma 3.5.2 (Negativity of contraction). Let $\pi: Y \to X$ be a proper birational morphism of normal quasi-projective varieties.

(1) If $Y$ is $\mathbb{Q}$-factorial and $E > 0$ is an exceptional $\mathbb{R}$-divisor, then there is a component $F$ of $E$ which is covered by curves $\Sigma$ such that $E \cdot \Sigma < 0$.

(2) If $\pi^*L = M + G + E$, where $L$ is an $\mathbb{R}$-Cartier divisor on $X$, $M$ is a $\pi$-nef $\mathbb{R}$-Cartier divisor on $Y$, $G \geq 0$, $E$ is $\pi$-exceptional, and $G$ and $E$ have no common components, then $E \geq 0$. Further if $E_i$ is a component of $E$ such that there is a component $E_j$ of $E$ with the same centre on $X$ as $E_i$, with the restriction of $M$ to $E_j$ not numerically $\pi$-trivial, then the coefficient of $E_i$ is strictly positive.

Proof. Cutting by hyperplanes in $X$, we reduce to the case when $X$ is a surface, in which case (1) reduces to the Hodge Index Theorem. (2) follows easily from (1), see for example (2.19) of [28]. □

Lemma 3.5.3. Let $\phi: X \dasharrow Y$ be a proper birational map of normal quasi-projective varieties, whose inverse does not contract any divisors, and let $D$ and $D'$ be $\mathbb{R}$-Cartier divisors such that $D' = \phi_*D$ is nef.

Then $\phi$ is $D$-non-positive (respectively $D$-negative) if we may write

$$D = (\phi^{-1})_*D' + E,$$

where $E \geq 0$ is $\phi$-exceptional (respectively $E \geq 0$ is $\phi$-exceptional and the support of $E$ contains the union of all $\phi$-exceptional divisors).

Proof. Easy consequence of (3.5.2). □
Definition 3.5.4. Let \( \pi: X \to U \) be a projective morphism of normal quasi-projective varieties. Suppose that \( K_X + \Delta \) is log canonical and let \( \phi: X \dasharrow Y \) be a birational map over \( U \) such that \( \phi^{-1} \) does not contract any divisors. Set \( \Gamma = \phi_\ast \Delta \).

- \( Y \) is a weak log canonical model for \( K_X + \Delta \) over \( U \) if \( \phi \) is \((K_X + \Delta)\)-non-positive and \( K_Y + \Gamma \) is nef over \( U \).
- \( Y \) is a log canonical model for \( K_X + \Delta \) if \( \phi \) is \((K_X + \Delta)\)-non-positive and \( K_Y + \Gamma \) is ample over \( U \).
- \( Y \) is a log terminal model for \( K_X + \Delta \) if \( \phi \) is \((K_X + \Delta)\)-negative, \( K_Y + \Gamma \) is divisorially log terminal and nef over \( U \), and \( Y \) is \( \mathbb{Q} \)-factorial.

Now suppose that \( \psi: X \dasharrow Z \) is a rational map.

- \( Z \) is an ample model for \( K_X + \Delta \) over \( U \) if there is a log terminal model \( \phi: X \dasharrow Y \) for \( K_X + \Delta \) over \( U \), a morphism \( f: Y \to Z \) over \( U \) and a divisor \( H \) on \( Z \), which is ample over \( U \), such that \( K_Y + \Gamma = f_\ast H \), where \( \Gamma = \phi_\ast \Delta \).

Lemma 3.5.5. Let \( \pi: X \to U \) be a proper morphism of normal quasi-projective varieties. Let \( (X, \Delta) \) be a kawamata log terminal pair. Let \( f: Z \to X \) be any log resolution of \( (X, \Delta) \) and suppose that we write

\[
K_Z + \Phi_0 = f^\ast (K_X + \Delta) + E,
\]

where \( \Phi_0 \) and \( E \) are effective, with no common components, \( f_\ast \Phi_0 = \Delta \) and \( E \) is exceptional. Let \( F \) be any effective divisor whose support is equal to the exceptional locus.

Then we may find \( \eta > 0 \) such that if \( \Phi = \Phi_0 + \eta F \) then

- \( f_\ast \Phi = \Delta \),
- \( K_Z + \Phi \) is kawamata log terminal,
- if \( \Delta \) is big over \( U \) then so is \( \Phi \), and
- the log terminal models (respectively weak log canonical models) over \( U \) of \( K_X + \Delta \) and the log terminal models (respectively weak log canonical models) over \( U \) of \( K_Z + \Phi \) are the same.

Proof. Everything is clear, apart from the fact that if \( \phi: Z \dasharrow W \) is a log terminal model (respectively weak log canonical model) over \( U \) of \( K_Z + \Phi \) then it is a log terminal model (respectively weak log canonical model) over \( U \) of \( K_X + \Delta \).

Let \( \psi: X \dasharrow W \) be the induced birational map and set \( \Psi = \phi_\ast \Phi \). By what we have already observed, possibly blowing up more, we may assume that \( \phi \) is a morphism. By assumption if we write

\[
K_Z + \Phi = \phi^\ast (K_W + \Psi) + G,
\]
then $G > 0$ and the support of $G$ is the full $\phi$-exceptional locus (respectively $G$ is exceptional). Thus

$$f^*(K_X + \Delta) + E + \eta F = \phi^*(K_W + \Psi) + G.$$  

By negativity of contraction (cf. (3.5.2)) $G - E - \eta F \geq 0$, so that in particular $\phi$ must contract every $f$-exceptional divisor and $\psi^{-1}$ does not contract any divisors. But then, $\psi$ is a log terminal model (respectively weak log canonical model) over $U$ by (3.5.3). □

**Lemma 3.5.6.** Let $\pi: X \to U$ be a projective morphism of normal quasi-projective varieties. Suppose that $K_X + \Delta$ is divisorial log terminal.

Then the ample model of $K_X + \Delta$ over $U$ is unique, if it exists at all.

**Proof.** Let $Z_1$ and $Z_2$ be ample models for $K_X + \Delta$ over $U$. By definition we may find weak log terminal models $\phi_i: X \to Y_i$ for $K_X + \Delta$ over $U$ and morphisms $f_i: Y_i \to Z_i$ over $U$ such that, $K_{Y_i} + \Gamma_i = f_i^*H_i$, where $H_i$ is an ample divisor over $U$ and $\Gamma_i = \phi_i^*\Delta$, for $i = 1$ and 2.

Suppose that $g: W \to X$ resolves the indeterminacy of $\phi_i$, for $i = 1$ and 2. Let $\Psi \geq 0$ be the divisor on $W$ whose existence is guaranteed by (3.5.5). Then $(Y_i, \Gamma_i)$ is also a log terminal model for $K_W + \Psi$, for $i = 1$ and 2.

Thus replacing $(X, \Delta)$ by $(W, \Psi)$ we may assume that $\phi_i$ is a morphism over $U$. In particular, there are effective divisors $E_i$, exceptional for $\phi_i$, such that

$$K_X + \Delta = \phi_i^*(K_{Y_i} + \Gamma_i) + E_i,$$

for $i = 1$ and 2. As $K_{Y_i} + \Gamma_i$ is nef, for $i = 1$ and 2, negativity of contraction implies that $E_1 = E_2$. In particular $g_1^*H_1 = g_2^*H_2$, where $g_i = f_i \circ \phi_i$ and so $Z_1 \simeq Z_2$. □

### 3.6. Convex geometry and Diophantine Approximation

**Definition 3.6.1.** Let $V$ be a real affine space. A **polytope** $P$ in $V$ is the convex hull of any finite set of points.

If $C$ is a convex subset of $V$ then we say that $v \in C$ is an **extreme point** if whenever

$$v = \sum r_i v_i,$$

where $r_1, r_2, \ldots, r_k$ are real numbers such that $\sum r_i = 1$, $r_i \geq 0$ and $v_1, v_2, \ldots, v_k$ belong to $C$ then $v = v_i$ for some $i$.

We say that $V$ is **defined over the rationals**, if $V = V' \otimes \mathbb{R}$, where $V'$ is a rational affine space. We say that a polytope $P$ is **rational** if the extreme points of $P$ are rational.
**Proposition 3.6.2.** Let $X$ be a normal quasi-projective variety, and let $V$ be a finite dimensional affine subspace of the real vector space of Weil divisors which is defined over the rationals.

Then $\mathcal{L}$ (cf. [1.1.4] for the definition) is a rational polytope.

**Proof.** Note that the set of divisors $\Delta$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier forms an affine subspace $W$ of $V$, which is defined over the rationals, so that, replacing $V$ by $W$, we may assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier for every $\Delta \in V$.

Let $\pi: Y \rightarrow X$ be a resolution of $X$, which is a log resolution of the support of any element of $V$. Given any divisor $\Delta \in V$, we may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta).$$

Then the coefficients of $\Gamma$ are rational affine linear functions of the coefficients of $\Delta$. On the other hand the condition that $K_X + \Delta$ is log canonical is equivalent to the condition that no component of $\Gamma$ has coefficient greater than one, and every coefficient of $\Delta$ is non-negative. \hfill \Box

**Lemma 3.6.3.** Let $\pi: X \rightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $C \subset \mathcal{L}_A \subset V_A$ be a rational simplex. Suppose that either

1. $A$ is big over $U$ and for every $\Delta \in C$, $K_X + \Delta$ is Kawamata log terminal, or

2. $A$ is ample over $U$, with no component in common with any element of $V$ and there is a divisor $\Delta_0'$ such that $K_X + \Delta_0'$ is Kawamata log terminal.

Then there is a divisor $A'$, ample and general over $U$ and a rational affine linear map

$$L: V_A \rightarrow V_A', \quad \text{where} \quad \Delta' = A' + B' = L(\Delta),$$

such that for all $\Delta \in C$,

- $K_X + \Delta' \sim_{\mathbb{Q}, U} K_X + \Delta$, and
- $K_X + \Delta'$ is Kawamata log terminal,

where $V_A'$ is the image of $V_A$.

**Proof.** We may assume that $\Delta = A + B$ where $A$ is a $\mathbb{Q}$-divisor and $B \geq 0$. Suppose that $A$ is big over $U$. Then there is a $\mathbb{Q}$-divisor $C$, which is ample over $U$, such that

$$A \sim_{\mathbb{Q}, U} C + C',$$
where $C' \geq 0$. By compactness, we may pick a rational number $\epsilon > 0$ such that

$$K_X + \Delta' = K_X + \epsilon C + (1 - \epsilon) \Delta + \epsilon (B + C') \sim_{Q,U} K_X + \Delta,$$

is kawamata log terminal for every $\Delta \in C$. Set $A' = \epsilon C$, $B' = (1 - \epsilon) \Delta + \epsilon (B + C')$ and let $L$ be the transformation

$$L(\Theta) = \Theta + \epsilon (C' - A) + A'.$$

Now suppose that $A$ is ample over $U$. Let $W$ be the vector space spanned by the components of $\Delta$ and $\Delta_0$. Since $L \subseteq W$ is a rational polytope, we may find a divisor $\Delta_0$ such that $K_X + \Delta_0$ is kawamata log terminal and $K_X + \Delta_0$ is $Q$-Cartier.

Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be the vertices of $C$. Pick a rational number $\epsilon > 0$ such that

$$A'_i = \epsilon (\Delta_i - \Delta_0) + (1/2 - \epsilon) A,$$

is ample. Pick $k_i$ such that $k_i A'_i$ is very ample, and let $k_i A_i$ be a very general element of the linear series $|k_i A'_i|$. Define a linear map $L$ by setting

$$L(\Delta_i) = (1 - \epsilon) \Delta_i - (1/2 - \epsilon) A + \epsilon \Delta_0 + A_i,$$

and extending by linearity. Let $A' = A/2$. \hfill $\square$

**Lemma 3.6.4.** Let $\Lambda \subset V$ be a lattice in a finite dimensional real vector space. Suppose that $v \in V$ is a vector

Then the set

$$X = \{ mv + \lambda \mid m \in \mathbb{N}, \lambda \in \Lambda \},$$

is dense in the smallest rational affine subspace $W$ of $V$ containing $v$.

**Proof.** We may assume that $W = V$. Let

$$q: V \longrightarrow V/\Lambda,$$

be the quotient map and let $G$ be the closure of the image of $X$. As $G$ is infinite and $V/\Lambda$ is compact, $G$ has an accumulation point. It follows that zero is also an accumulation point, so that $G$ is a closed subgroup.

Let $G_0$ be the connected component of the identity. Then $G_0$ is a Lie subgroup of $V/\Lambda$ and the tangent space to $G_0$ is $q(W)$, where $W$ may be identified with the tangent space to $G_0$ at the identity. Then $W$ is a rational subspace of $\mathbb{R}^n$. On the other hand, $G/G_0$ is finite as it is discrete and compact. Thus a translate of $v$ by a rational vector is contained in $W$ and so $W = V$. \hfill $\square$

**Lemma 3.6.5.** Let $C$ be a rational polytope contained in a vector space $V$ of dimension $n$, defined over the rationals. Fix a positive integer $k$ and a positive real number $\alpha$.  

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If \( v \in \mathcal{C} \) then we may find vectors \( v_1, v_2, \ldots, v_p \in \mathcal{C} \) and positive integers \( m_1, m_2, \ldots, m_p \), which are divisible by \( k \), such that \( v \) is a convex linear combination of the vectors \( v_1, v_2, \ldots, v_p \) and

\[
\|v_i - v\| < \frac{\alpha}{m_i} \quad \text{where} \quad m_i v_i \text{ is integral.}
\]

**Proof.** Rescaling by \( k \), we may assume that \( k = 1 \). We may assume that \( v \) is not contained in any proper rational affine linear subspace. In particular, as \( \mathcal{C} \) is a rational polytope, it has rational faces and so \( v \) is contained in the interior of \( \mathcal{C} \).

After fixing a suitable basis for \( V \) and possibly shrinking \( \mathcal{C} \), we may assume that \( \mathcal{C} = [0, 1]^n \subseteq \mathbb{R}^n \) and \( v = (x_1, x_2, \ldots, x_n) \in (0, 1)^n \), where the numbers \( x_1, x_2, \ldots, x_n \) are independent over the rationals. By [3.6.4], for each subset \( I \subseteq \{1, 2, \ldots, n\} \), there exist \( v_I = (s_1, s_2, \ldots, s_n) \in (0, 1)^n \cap \mathbb{Q}^n \) and an integer \( m_I \), such that \( m_I v_I \) is integral, and

\[
\|v - v_I\| < \frac{\alpha}{m_I} \quad \text{and} \quad s_j < x_j \quad \text{if and only if} \quad j \in I.
\]

In particular \( v \) is contained inside the rational polytope \( \mathcal{B} \subseteq \mathcal{C} \) generated by the \( v_I \). Thus \( v \) is a convex linear combination of a subset \( v_1, v_2, \ldots, v_p \) of the extreme points of \( \mathcal{B} \). \( \square \)

### 3.7. Stable base locus

We need to extend the definition of the stable base locus to the case of a real divisor:

**Definition 3.7.1.** Let \( D \) be an \( \mathbb{R} \)-divisor on a locally Noetherian separated scheme, regular in codimension one. The **real linear system** associated to \( D \) is

\[
|D|_\mathbb{R} = \{ C \geq 0 \mid C \sim_\mathbb{R} D \}.
\]

The **stable base locus** of \( D \) is the intersection of the elements of the real linear system \( |D|_\mathbb{R} \). The **stable fixed divisor** is the divisorial support of the stable base locus.

**Remark 3.7.2.** The stable base locus and the stable fixed divisor are only defined as closed subsets, they do not have any scheme structure.

**Lemma 3.7.3.** Let \( D \) be an integral Weil divisor on a normal variety \( X \).

Then the stable base locus as defined in [3.7.1] coincides with the usual definition of the stable base locus.

**Proof.** Let

\[
|D|_\mathbb{Q} = \{ C \geq 0 \mid C \sim_\mathbb{Q} D \}.
\]
Let $R$ be the intersection of the elements of $|D|_\mathbb{R}$ and let $Q$ be the intersection of the elements of $|D|_\mathbb{Q}$. It suffices to prove that $Q = R$. As $|D|_\mathbb{Q} \subset |D|_\mathbb{R}$, it is clear that $R \subset Q$.

Suppose that $x \notin R$. We want to show that $x \notin Q$. Replacing $X$ by the blow up of $X$ at $x$, we may assume that there is a divisor $C$ not a component of $R$ and it suffices to prove that $C$ is not a component of $Q$. We may find $D' \in |D|_\mathbb{R}$ such that $C$ is not a component of $D'$. But then

$$D = D' + \sum r_i(f_i),$$

where $f_i$ are rational functions on $X$ and $r_i$ are real numbers. Let $V$ be the real subspace of the vector space of all Weil divisors on $X$ spanned by the components of $D$, $D'$ and $(f_i)$ and let $W$ be the span of the $(f_i)$. Then $W$ and $V$ are defined over the rationals. Set

$$\mathcal{P} = \{ D'' \in V \mid D'' \geq 0, \text{ mult}_C(D'') = 0, D'' - D \in W \} \subset |D|_\mathbb{R}.$$

Then $\mathcal{P}$ is a rational polyhedron. As $D' \in \mathcal{P}$, $\mathcal{P}$ is non-empty, and so it must contain a rational point $D''$. We may write

$$D'' = D + \sum s_i(f_i),$$

where $s_i$ are real numbers. Since $D''$ and $D$ have rational coefficients, it follows that we may find $s_i$ which are rational. But then $D'' \in |D|_\mathbb{Q}$, and so $C$ is not a component of $D''$, nor therefore of $Q$. \hfill $\square$

**Proposition 3.7.4.** Let $X$ be a normal projective variety and let $D \geq 0$ be an $\mathbb{R}$-divisor. Then we may find $\mathbb{R}$-divisors $M$ and $F$ such that

1. $M \geq 0$ and $F \geq 0$,
2. $D \sim_\mathbb{R} M + F$,
3. every component of $F$ is a component of the stable fixed divisor, and
4. if $B$ is a component of $M$ then some multiple of $B$ is mobile.

We need two basic results:

**Lemma 3.7.5.** Let $X$ be a normal variety and let $D$ and $D'$ be two $\mathbb{R}$-divisors such that $D \sim_\mathbb{R} D'$.

Then we may rational functions $f_1, f_2, \ldots, f_k$ and real numbers $r_1, r_2, \ldots, r_k$ which are independent over the rationals such that

$$D = D' + \sum r_i(f_i).$$

In particular every component of $(f_i)$ is either a component of $D$ or of $D'$. 

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Proof. By assumption we may find \( f_1, f_2, \ldots, f_k \) and real numbers \( r_1, r_2, \ldots, r_k \) such that

\[
D = D' + \sum_{i=1}^{k} r_i(f_i).
\]

Pick \( k \) minimal with this property. Suppose that the real numbers \( r_i \) are not independent over \( \mathbb{Q} \). Then we can find rational numbers \( d_i \), not all zero, such that

\[
\sum_{i} d_ir_i = 0.
\]

Possibly re-ordering we may assume that \( d_k \neq 0 \). Multiplying through by an integer we may assume that \( d_i \in \mathbb{Z} \). Possibly replacing \( f_i \) by \( f_i^{-d_i} \) we may assume that if \( d_i \neq 0 \) then \( d_i = d \). For \( 1 \leq i < k \), set

\[
g_i = \begin{cases} 
  f_i/f_k & \text{if } d_i = d \\
  f_i & \text{if } d_i = 0.
\end{cases}
\]

Then

\[
D = D' + \sum_{i=1}^{k-1} r_i(g_i),
\]

which contradicts our choice of \( k \).

Now suppose that \( B \) is a component of \( (f_i) \). Then

\[
\text{mult}_B(D) = \text{mult}_B(D') + \sum r_j n_j,
\]

where \( n_j = \text{mult}_B(f_j) \) is an integer and \( n_i \neq 0 \). But then one of \( \text{mult}_B(D) - \text{mult}_B(D') \) must be non zero. \( \square \)

Lemma 3.7.6. Let \( D' \sim D \) be two effective \( \mathbb{R} \)-divisors on a variety \( X \) with no common components.

Then we may find \( D'' \in |D|_{\mathbb{R}} \) such that every component of \( D'' \) has a multiple which is mobile.

Proof. We may write

\[
D' = D + \sum r_i(f_i) = D + R,
\]

where \( r_i \in \mathbb{R} \) and \( f_i \) are rational functions on \( X \). By (3.7.5) we may assume that every component of \( R \) is a component of \( D + D' \).

We proceed by induction on the number of components of \( D + D' \).

Pick positive rational numbers \( q_i \) and let \( S = \sum q_i(f_i) \). Then we may find \( C \geq 0 \) and \( C' \geq 0 \) with no common components such that

\[
C' = C + S.
\]

Lemma 3.7.6. Let \( D' \sim D \) be two effective \( \mathbb{R} \)-divisors on a variety \( X \) with no common components.

Then we may find \( D'' \in |D|_{\mathbb{R}} \) such that every component of \( D'' \) has a multiple which is mobile.

Proof. We may write

\[
D' = D + \sum r_i(f_i) = D + R,
\]

where \( r_i \in \mathbb{R} \) and \( f_i \) are rational functions on \( X \). By (3.7.5) we may assume that every component of \( R \) is a component of \( D + D' \).

We proceed by induction on the number of components of \( D + D' \).

Pick positive rational numbers \( q_i \) and let \( S = \sum q_i(f_i) \). Then we may find \( C \geq 0 \) and \( C' \geq 0 \) with no common components such that

\[
C' = C + S.
\]
If we pick $q_i$ sufficiently close to $r_i$ we may assume that $C$ is supported on $D$ and $C'$ is supported on $D'$. We have that $mC \sim mC'$ for some integer $m > 0$. By Bertini we may find $C'' \in |C|$ such that every component of $C''$ has a multiple which is mobile. Pick $\lambda > 0$ maximal such that $D_1 = D - \lambda C \geq 0$ and $D'_1 = D' - \lambda C' \geq 0$. Note that

$$D_1 \sim_d D'_1$$

and that $D_1 + D'_1$ has fewer components than $D + D'$. By induction we may then find

$$D'' \in |D_1|_R,$$

such that every component of $D''$ has a multiple which is mobile. But then

$$D'' = \lambda C'' + D''_1 \in |D|_R,$$

and every component of $D''$ has a multiple which is mobile.

□

Proof of (3.7.4). We may write $D \sim_d M + F$, where every component of $F$ is a stable fixed divisor and no component of $M$ is a stable fixed divisor. We proceed by induction on the number of components $B$ of $M$ such that no multiple of $B$ is mobile.

Fix one such component $B$. It suffices to find $D' \in |D|_R$ such that $B$ is not a component of $D'$ and with the property that if $B'$ is a component of $D'$ such that no multiple of $B'$ is mobile, then $B'$ is also a component of $D$. Now we may find $D_1 \in |D|_R$ such that $B$ is not a component of $D_1$. Cancelling common components of $D_1$ and $D$, by (3.7.6), one sees that $D \sim_d D' = D'' + E$ where every component of $D''$ has a multiple which is mobile and Supp($E$) $\subset$ Supp($D$) $\cap$ Supp($D_1$).

Lemma 3.7.7. Let $\pi: X \longrightarrow U$ be a birational morphism of normal $\mathbb{Q}$-factorial projective varieties, let $D$ be an $\mathbb{R}$-Cartier divisor on $U$ such that $|D|_R \neq \emptyset$. Let $E$ be an effective $\mathbb{R}$-Cartier divisor on $X$ such that $\pi_* E$ is supported on the stable fixed divisor of $D$.

If $G \in |\pi^* D + E|_R$ then $G - E \geq 0$. In particular the stable fixed divisor of $\pi^* D$ is contained in the stable fixed divisor of $\pi^* D + E$.

Proof. We may assume that $D \geq 0$ and cancelling we may assume that no component of $E$ is a component of $G$. It suffices to prove that $E = 0$. Suppose not.

By induction we may assume that $E$ is irreducible. Rescaling we may assume that $E$ is a prime divisor. If $E$ is exceptional then $E$ is covered by a family of curves $\Sigma$ such that $E \cdot \Sigma < 0$. But then $G \cdot \Sigma < 0$ so that $\text{mult}_E G > 0$, a contradiction.
Otherwise $E$ is in the stable fixed divisor of $\pi^*D$. Hence we may assume that $\pi$ is the identity. Let $k = \text{mult}_E D$. Then $k > 0$ as $D \geq 0$ and $E$ is a component of the stable fixed divisor. Set

$$C = D + k(G - E).$$

By construction $\text{mult}_E C = 0$, so that $C \geq 0$. On the other hand

$$C \sim_{\mathbb{R}} (k + 1)D,$$

so that $\text{mult}_E C > 0$ by definition of the stable fixed divisor, a contradiction. \[\square\]

3.8. **Rational curves of low degree.** We will need the following generalisation of a result of Kawamata, see Theorem 1 of [22], which is proved by Shokurov in the appendix to [35]:

**Theorem 3.8.1.** Let $\pi : X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose that $(X, \Delta)$ is a log canonical pair of dimension $n$, where $K_X + \Delta$ is $\mathbb{R}$-Cartier. Suppose that there is a divisor $\Delta_0$ such that $K_X + \Delta_0$ is kawamata log terminal.

If $R$ is an extremal ray of $\mathcal{NE}(X/U)$ that is $(K_X + \Delta)$-negative, then there is a rational curve $\Sigma$, contracted by $\pi$ and spanning $R$, such that

$$0 < -(K_X + \Delta) \cdot \Sigma \leq 2n.$$

**Proof.** Passing to an open subset of $U$, we may assume that $U$ is affine. Let $V$ be the rational space spanned by the components of $\Delta + \Delta_0$. By (3.6.2) the space $\mathcal{L}$ of log canonical divisors is a rational polytope. Since $\Delta_0 \in \mathcal{L}$, we may find $\mathbb{Q}$-divisors $\Delta_i$ with limit $\Delta$, such that $K_X + \Delta_i$ is kawamata log terminal.

In particular we may assume that $(K_X + \Delta_0) \cdot R < 0$. Replacing $\pi$ by the contraction defined by the extremal ray $R$, we may assume that $-(K_X + \Delta)$ is $\pi$-ample.

Theorem 1 of [22] implies that we can find a rational curve $\Sigma_i$ contracted by $\pi$ such that

$$-(K_X + \Delta_i) \cdot \Sigma_i \leq 2n.$$

Pick a $\pi$-ample $\mathbb{Q}$-divisor $A$ such that $-(K_X + \Delta + A)$ is also $\pi$-ample. Now

$$A \cdot \Sigma_i = (K_X + \Delta_i + A) \cdot \Sigma_i - (K_X + \Delta_i) \cdot \Sigma_i < 2n.$$

It follows that the curves $\Sigma_i$ belong to a bounded family. Thus, possibly passing to a subsequence, we may assume that $\Sigma = \Sigma_i$ is constant. In this case

$$-(K_X + \Delta) \cdot \Sigma = \lim_{i} -(K_X + \Delta_i) \cdot \Sigma \leq 2n. \quad \square$$
Corollary 3.8.2. Let $\pi: X \rightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose the pair $(X, \Delta = A + B)$ has log canonical singularities, where $A$ is ample over $U$ and $B \geq 0$. Suppose that there is a divisor $\Delta_0$ such that $K_X + \Delta_0$ is Kawamata log terminal.

Then there are only finitely many $(K_X + \Delta)$-negative extremal rays $R_1, R_2, \ldots, R_k$ of $\overline{\text{NE}}(X/U)$.

Proof. Let $R$ be a $(K_X + \Delta)$-negative extremal ray of $\overline{\text{NE}}(X/U)$. Then

$$-(K_X + B) \cdot R = -(K_X + \Delta) \cdot R + A \cdot R > 0.$$ 

By (3.8.1) $R$ is spanned by a curve $\Sigma$ such that

$$-(K_X + B) \cdot \Sigma \leq 2n.$$ 

But then

$$A \cdot \Sigma = -(K_X + B) \cdot \Sigma + (K_X + \Delta) \cdot \Sigma \leq 2n.$$ 

$\square$

3.9. The MMP with scaling. To run the MMP with scaling, we will need the following key result:

Lemma 3.9.1. Let $\pi: X \rightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose that the pair $(X, \Delta = A + B)$ has Kawamata log terminal singularities, where $A \geq 0$ is big over $U$, $B \geq 0$, $D$ is a nef $\mathbb{R}$-Cartier divisor over $U$, but $K_X + \Delta$ is not nef over $U$.

Set

$$\lambda = \sup \{ \mu | D + \mu(K_X + \Delta) \text{ is nef over } U \}.$$ 

Then there is a $(K_X + \Delta)$-negative extremal ray $R$ over $U$, such that

$$(D + \lambda(K_X + \Delta)) \cdot R = 0.$$ 

Proof. By (3.6.3) we may assume that $A$ is ample over $U$.

By (3.8.2) there are only finitely many $(K_X + \Delta)$-negative extremal rays $R_1, R_2, \ldots, R_k$ over $U$. For each $(K_X + \Delta)$-negative extremal ray $R_i$, pick a curve $\Sigma_i$ which generates $R_i$. Let

$$\mu = \min_i \frac{D \cdot \Sigma_i}{-(K_X + \Delta) \cdot \Sigma_i}.$$ 

Then $D + \mu(K_X + \Delta)$ is nef over $U$, since it is non-negative on each $R_i$, but it is zero on one of the extremal rays $R = R_i$. Thus $\lambda = \mu$. $\square$

Lemma 3.9.2. Let $\pi: X \rightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose the pair $(X, \Delta = A + B)$ has Kawamata log terminal singularities, where $A$ is big over $U$, $B \geq 0$, and $C$ is an $\mathbb{R}$-Cartier divisor such that $K_X + \Delta$ is not nef over $U$, but $K_X + \Delta + C$ is nef over $U$ and log canonical.

Then there is a $(K_X + \Delta)$-negative extremal ray $R$ and a real number $0 < \lambda \leq 1$ such that $K_X + \Delta + \lambda C$ is nef over $U$ but trivial on $R$. 

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Proof. Apply (3.9.1) to $D = K_X + \Delta + C$. □

**Remark 3.9.3.** Assuming existence and termination of the relevant flips, we may use (3.9.2) to define a special minimal model program, which we will refer to as the $(K_X + \Delta)$-MMP with scaling of $C$.

We start with a $\mathbb{Q}$-factorial kawamata log terminal pair $(X, \Delta)$ and an effective $\mathbb{R}$-divisor $C$ such that $K_X + \Delta + C$ is nef and kawamata log terminal. We pick $R$ and $\lambda \geq 0$ as in Lemma 3.9.2 above. If $K_X + \Delta$ is nef we stop, otherwise $\lambda > 0$ and we let $f: X \rightarrow Z$ be the extremal contraction defined by $R$. If $f$ is not birational, we have a Mori fiber space and we stop. If $f$ is birational then either $f$ is divisorial and we replace $X$ by $Z$ or $f$ is small and assuming the existence of the flip $f^+: X^+ \rightarrow Z$, we replace $X$ by $X^+$. In either case $K_X + \Delta + \lambda C$ is nef and kawamata log terminal and so we may repeat the process.

We will need a result about the divisorially log terminal MMP:

**Lemma 3.9.4.** Let $\pi: X \rightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose that $K_X + \Delta$ is divisorially log terminal, $X$ is $\mathbb{Q}$-factorial and let $\phi: X \dashrightarrow Y$ be a sequence of steps of the $(K_X + \Delta)$-MMP. Let $\Gamma = \phi_*\Delta$.

Then

1. $\phi$ is an isomorphism at the generic point of every log canonical centre of $K_Y + \Gamma$.

2. If $K_X + \Delta \sim_{\mathbb{R}, U} K_X + \Delta' = K_X + S + A + B$, where $S = \Delta' \cdot \mathbb{1}$, $A$ is ample over $U$ and $B \geq 0$, then $K_Y + \Gamma \sim_{\mathbb{R}, U} K_Y + T + C + D$, where $\phi_* S = T = \Delta' \cdot \Gamma$, $C$ is ample over $U$, $D \geq 0$ and $K_Y + T + C + D$ is divisorially log terminal.

*Proof.* We first prove (1). We may assume that $\phi$ is either a flip or a divisorial contraction.

Let $p: W \rightarrow X$ and $q: W \rightarrow Y$ be a common log resolution, which resolves the indeterminancy of $\phi$. We may write

$$p^*(K_X + \Delta) = q^*(K_Y + \Gamma) + E,$$

where $E$ is exceptional and contains every exceptional divisor over the locus where $\phi^{-1}$ is not an isomorphism. In particular the log discrepancy of every valuation with centre on $Y$ contained in the locus where $\phi^{-1}$ is not an isomorphism with respect to $K_Y + \Gamma$ is strictly greater than the log discrepancy with respect to $K_X + \Delta$. Hence (1).

Now suppose that $K_X + \Delta \sim_{\mathbb{R}, U} K_X + S + A + B$. Pick a divisor $C$ in $Y$ which is ample over $U$. Possibly replacing $C$ by a smaller multiple, we may assume that $A - \phi^{-1}_* C$ is ample over $U$. Thus $A - \phi^{-1}_* C$ is $\mathbb{R}$-linearly equivalent to an $\mathbb{R}$-divisor $A'$ which does not contain any log
canonical centre of \((X, \Delta')\). Note that for \(0 < \epsilon \ll 1\), we may assume that \(\phi\) is a flip or a divisorial contraction for \(K_X + S + (1 - \epsilon)A + \epsilon A' + B\) which is also divisorially log terminal. Thus by (1) \(\phi_* A - C\) is \(\mathbb{R}\)-linearly equivalent to \(\phi_* A'\) which does not contain any log canonical centre of \((Y, \Gamma)\). (2) is then clear.

\[\Box\]

3.10. **Shokurov’s Polytopes.** We will need some results from [38].

First some notation. Given a ray \(R \subset \text{NE}(X)\), let

\[R^\perp = \{ \Delta \in L \mid (K_X + \Delta) \cdot R = 0 \}.\]

**Theorem 3.10.1.** Let \(\pi : X \rightarrow U\) be a projective morphism of normal quasi-projective varieties. Let \(V\) be a finite dimensional affine subspace of the real vector space of Weil divisors on \(X\), which is defined over the rationals. Fix \(A\), an ample divisor, general over \(U\). Suppose that there is a divisor \(\Delta_0\) such that \(K_X + \Delta_0\) is kawamata log terminal.

Then the set of hyperplanes \(R^\perp\) is finite in \(L\), as \(R\) ranges over the set of extremal rays of \(\text{NE}(X/U)\). In particular \(N_A\) is a rational polytope.

**Corollary 3.10.2.** Let \(\pi : X \rightarrow U\) be a projective morphism of normal quasi-projective varieties. Let \(V\) be a finite dimensional affine subspace of the real vector space of Weil divisors on \(X\), which is defined over the rationals. Fix \(A\), an ample \(\mathbb{Q}\)-Cartier divisor, general over \(U\). Let \(\phi : X \rightarrow Y\) be any birational map over \(U\), whose inverse does not contract any divisors. Suppose that there is a divisor \(\Delta_0\) such that \(K_X + \Delta_0\) is kawamata log terminal.

Then \(P_Y\) is a rational polytope and there are finitely many morphisms \(f_j : Y \rightarrow Z_j\) such that for any \(\Delta \in P_Y\), there is an index \(j\) and an ample divisor \(H\) on \(Z_j\) such that \(K_Y + \Gamma = \phi_*(K_X + \Delta) = f_j^*H\).

**Corollary 3.10.3.** Let \(\pi : X \rightarrow U\) be a projective morphism of normal quasi-projective varieties. Let \(V\) be a finite dimensional affine subspace of the real vector space of Weil divisors on \(X\), which is defined over the rationals. Fix \(A\), an ample \(\mathbb{Q}\)-divisor, general over \(U\). Let \((X, \Delta)\) be a kawamata log terminal pair, where \(\Delta \in V\) and let \(f : X \rightarrow Z\) a morphism over \(U\) such that \(K_X + A + \Delta = f^*H\), for some ample divisor \(H\). Let \(\phi : X \rightarrow Y\) be a birational map over \(Z\).

Then there is a positive rational number \(\epsilon > 0\) such that for all \(\Delta' \in V\) with \(\|\Delta' - \Delta\| \leq \epsilon\) then \(\phi : X \rightarrow Y\) is a log terminal model of \(K_X + A + \Delta'\) over \(Z\) iff it is a log terminal model of \(K_X + A + \Delta'\) over \(U\).

**Proof of (3.10.1).** Since \(L\) is compact it suffices to prove this locally about any point \(\Delta \in L\). By (3.6.3) we may assume that \(K_X + \Delta\)
is kawamata log terminal. Fix $\epsilon > 0$ such that if $\|\Delta' - \Delta\| < \epsilon$, then $\Delta' - \Delta + A/2$ is ample. Let $R$ be an extremal ray $U$ such that 
\[(K_X + \Delta') \cdot R = 0, \text{ where } \|\Delta' - \Delta\| < \epsilon.\]
We have
\[(K_X + \Delta - A/2) \cdot R = (K_X + \Delta') \cdot R - (\Delta' - \Delta + A/2) \cdot R < 0.\]
Finiteness then follows from (3.8.2).

$\mathcal{N}_A$ is surely a closed subset of $\mathcal{L}_A$. If $K_X + \Delta$ is not nef over $U$ then (3.8.1) implies that $K_X + \Delta$ is negative on a rational curve $\Sigma$ which generates an extremal ray $R$ of $\overline{\text{NE}}(X/U)$. Thus $\mathcal{N}_A$ is the intersection of $\mathcal{L}_A$ with the finitely many half-spaces determined by the extremal rays of $\overline{\text{NE}}(X/U)$.

**Proof of (3.10.2).** We may as well suppose that $V$ is the space spanned by the components of a divisor $D$. By compactness, it is enough to show that $\mathcal{P}_Y$ is locally a rational polytope.

Let $\Delta \in \mathcal{P}_Y$. By (2) of (3.6.3) we may assume that $K_X + \Delta$ is kawamata log terminal, so that $K_Y + \Gamma$ is kawamata log terminal as well. Let $C = \phi_*A$. Then $C$ is big over $U$. By (1) of (3.6.3) and (3.10.1) we may assume that $\mathcal{N}_C$ is locally a rational polytope about $\Gamma$.

Let $p: W \to X$ be a log resolution of $(X, \Delta)$ which resolves the indeterminacy locus of $\phi$, via a birational map $q: W \to Y$. We may write
\[
K_W + \Psi = p^*(K_X + \Delta)
\]
\[
K_W + \Phi = q^*(K_Y + \Gamma).
\]

Note that $\Delta \in \mathcal{P}_Y$ iff $\Gamma = \phi_*\Delta \in \mathcal{N}_C$ and $\Psi - \Phi \geq 0$. Since the assignment $\Delta \to \Gamma$ is rational linear, the first statement is clear.

Let $\Delta_k, k = 1$ and 2, belong to the interior of the same face of $\mathcal{P}_Y$. Then $K_Y + \Gamma_k = \phi_*(K_X + \Delta_k)$ are zero on precisely the same curves, for $k = 1, 2$ and the ample model is determined by exactly these curves.

**Proof of (3.10.3).** By (3.10.1) we may find finitely many extremal rays $R_1, R_2, \ldots, R_k$ of $Y$ over $U$ such that if $K_Y + \Gamma' = K_Y + \phi_*\Delta'$ is not nef over $U$, then it is negative on one of these rays. We may write
\[
K_Y + \Gamma' = K_Y + \Gamma + (\Gamma' - \Gamma) = f^*H + \phi_*(\Delta' - \Delta).
\]
Therefore for $\epsilon$ sufficiently small, if $K_Y + \Gamma'$ is not nef over $U$ it is negative on an extremal ray $R_i$, which is extremal over $Z$. In particular if $\phi$ is a log terminal model of $K_X + \Delta'$ over $Z$ then it is a log terminal model over $U$. 

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The other direction is clear.

\section*{4. Special termination with scaling}

In this section we show that we have special termination of the MMP with scaling.

\textbf{Lemma 4.1.} Assume Theorem $\mathbb{B}_n$ and Theorem $\mathbb{C}_n$.

Let $\pi_i : X_i \rightarrow U$ be a sequence of projective morphisms of normal quasi-projective varieties, where $X_i$ and $X_j$ are isomorphic in codimension one over $U$. Suppose that $K_{X_i} + \Delta_i$ is kawamata log terminal and nef over $U$. Suppose that there are fixed divisors $A_1$ and $B_1$ on $X = X_1$, with transforms $A_i$ and $B_i$ on $X_i$, such that

$$A_i \leq \Delta_i \leq B_i,$$

where $A_1$ is big over $U$ and $K_{X_1} + B_1$ is kawamata log terminal. Let $n$ be the dimension of $X$.

Then the set of isomorphism classes of birational maps

$$\{ X \rightarrow X_i \mid i \in \mathbb{N} \},$$

is finite.

Here, two birational maps $\phi : X \rightarrow X_i$ and $\psi : X \rightarrow X_j$ are isomorphic if there exists an isomorphism $\eta : X_i \rightarrow X_j$ such that $\psi = \eta \circ \phi$.

\textit{Proof.} As we are assuming Theorem $\mathbb{B}_n$, replacing $X$ by a log terminal model we may assume that $X$ is $\mathbb{Q}$-factorial.

Let $\Delta'_i$ be the strict transform of $\Delta_i$ on $X$. By (1) of (3.6.3) we may assume that $A_1$ is ample over $U$. As $(X_i, \Delta_i)$ is a weak log canonical model of $(X, \Delta'_i)$, the result follows as we are assuming Theorem $\mathbb{C}_n$.

\textbf{Lemma 4.2.} Assume Theorem $\mathbb{B}_{n-1}$ and Theorem $\mathbb{C}_{n-1}$.

Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties, where $X$ is a $\mathbb{Q}$-factorial variety of dimension $n$. Suppose that

$$K_X + \Delta + C = K_X + S + A + B + C,$$

is divisorially log terminal and nef over $U$, where $A$ is ample over $U$ and $B, C \geq 0$.

Then every sequence of flips over $U$ for the $(K_X + \Delta)$-MMP with scaling of $C$ is eventually disjoint from $S = \cup \Delta_j$. 

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Proof. Suppose not. Let \( X_i \rightarrow X_{i+1} \) be an infinite sequence of flips over \( U \), starting with \( X_1 := X \), for the \((K_X + \Delta)\)-MMP with scaling of \( C \), which meets \( S \) infinitely often. Then there is a component \( T \) of \( S \) that intersects the flipping locus infinitely many times. Pick a rational number \( \epsilon > 0 \) such that intersects the flipping locus infinitely many times. Pick a rational number \( \epsilon > 0 \) such that \( A' = A + \epsilon(S - T) \) is ample over \( U \). Replacing \( A' \) by an \( \mathbb{R} \)-linearly equivalent over \( U \) divisor, we may assume that

\[
K_X + S - \epsilon(S - T) + A' + B \sim_{\mathbb{R}, U} K_X + S + A + B,
\]
is purely log terminal. Note that every step of the \((K_X + \Delta)\)-MMP over \( U \) is a step of the \((K_X + S - \epsilon(S - T) + A' + B)\)-MMP over \( U \), and \( S - \epsilon(S - T) + A' + B \) is ample. Thus, replacing \( K_X + \Delta \) by \( K_X + S - \epsilon(S - T) + A' + B \) we may assume that \( S \) is irreducible and \( K_X + \Delta + C \) is purely log terminal. Let \( S_i \) be the strict transform of \( S \) in \( X_i \) and let \( S_i \rightarrow S_{i+1} \) be the induced birational map.

By [7], we may assume that \( X_i \rightarrow X_{i+1} \) is an isomorphism at the generic point of every divisor on \( S_i \) and \( S_{i+1} \). In particular we may assume that \( S_i \rightarrow S_{i+1} \) is an isomorphism in codimension one.

Let \( \Delta_i \) be the strict transform of \( \Delta \) on \( X_i \). By adjunction we may write

\[
(K_{X_i} + \Delta_i)|_{S_i} = K_{S_i} + \Theta_i.
\]

Then \( \Theta_i \) is the strict transform of \( \Theta_1 \). By definition of the MMP with scaling there is a \( t_i \geq 0 \) such that \( K_{S_i} + \Theta_i + t_iC_i|_{S_i} \) is nef and kawamata log terminal. Thus the hypotheses of (1.1) are satisfied, and the set of isomorphism classes

\[
\{ S \rightarrow S_i \mid i \in \mathbb{N} \},
\]
is finite, so that the set of pairs

\[
\{ (S_i, \Theta_i) \mid i \in \mathbb{N} \},
\]
is also finite.

On the other hand, let \( X_i \rightarrow Z_i \) be the flipping contraction and let \( T_i \) be the normalisation of the image of \( S_i \) in \( Z_i \), so that there are birational morphisms \( p_i : S_i \rightarrow T_i \) and \( q_i : S_{i+1} \rightarrow T_i \). Note that \( -(K_{S_i} + \Theta_i) \) is \( p_i \)-ample whilst \( (K_{S_{i+1}} + \Theta_{i+1}) \) is \( q_i \)-ample. By assumption infinitely often the flipping locus intersects \( S_i \). If \( p_i \) is an isomorphism and the flipping locus intersects \( S_i \), then \( S_i \cdot \Sigma_i > 0 \), where \( \Sigma_i \) is a flipping curve. But then \( S_{i+1} \) must intersect any flipped curve negatively, so that all flipped curves lie in \( S_{i+1} \) and \( q_{i+1} \) is not an isomorphism. In particular infinitely often one of the birational morphisms \( p_i \) or \( q_i \) is not an isomorphism.

Thus we may assume that \( p_i \) or \( q_i \) is not an isomorphism, where the isomorphism class of \( S_1 \) is repeated infinitely often. Pick any valuation
ν whose centre is contained in the locus where $S_1 \rightarrow S_2$ is not an isomorphism. By (2.28) of [28]

$$a(\nu, S_1, \Theta_1) < a(\nu, S_2, \Theta_2)$$

and

$$a(\nu, S_i, \Theta_i) \leq a(\nu, S_{i+1}, \Theta_{i+1}),$$

for all $i \geq 2$, a contradiction. □

We use [4.2] to run a special MMP:

**Lemma 4.3.** Assume Theorem [B]$_{n-1}$, Theorem [C]$_{n-1}$ and Theorem [A].

Let $\pi: X \rightarrow U$ be a projective morphism of normal quasi-projective varieties, where $X$ is $\mathbb{Q}$-factorial of dimension $n$. Suppose that $(X, \Delta + C = S + A + B + C)$ is a divisorially log terminal pair, such that $\Delta + C \downarrow = S$, $A$ is ample over $U$, and $B, C \geq 0$. Suppose that there is an $\mathbb{R}$-divisor $D \geq 0$ whose support is contained in $S$ and a real number $\alpha \geq 0$, such that

$$(*) \quad K_X + \Delta \sim_{\mathbb{R}, U} D + \alpha C.$$  

If $K_X + \Delta + C$ is nef over $U$ then there is a log terminal model $\phi: X \rightarrow Y$ for $K_X + \Delta$ over $U$.

**Proof.** Since $X$ is $\mathbb{Q}$-factorial, (2) of (3.6.3) implies that we may find $\Delta' \sim_{\mathbb{R}, U} \Delta$ such that $K_X + \Delta'$ is Kawamata log terminal. In particular we may apply (3.9.2) to $K_X + \Delta'$ to run the $(K_X + \Delta)$-MMP with scaling of $C$ over $U$.

We run the the $(K_X + \Delta)$-MMP over $U$ with scaling of $C$. Pick $t \in [0, 1]$ minimal such that $K_X + \Delta + tC$ is nef over $U$. If $t = 0$ we are done. Otherwise we may find a $(K_X + \Delta)$-negative extremal ray $R$ over $U$, such that $(K_X + \Delta + tC) \cdot R = 0$. Let $f: X \rightarrow Z$ be the associated contraction over $U$. As $t > 0$, $C \cdot R > 0$ and so $D \cdot R < 0$. In particular $f$ is always birational.

If $f$ is divisorial, then we can replace $X, S, A, B, C$ and $D$ by their images in $Z$. Note that $(*)$ continues to hold.

Otherwise $f$ is small. As $D \cdot R < 0$, $R$ is spanned by a curve $\Sigma$ which is contained in a component $T$ of $S$, where $T \cdot \Sigma < 0$. Note that $K_X + S + A + B - \epsilon(S - T)$ is a purely log terminal divisor for any positive $\epsilon \ll 1$, and so $f$ is a pl-flip.

As we are assuming Theorem [A], the flip $f': X' \rightarrow Z$ of $f: X \rightarrow Z$ exists. Again, if we replace $X, S, A, B, C$ and $D$ by their images in $X'$, then $(*)$ continues to hold. On the other hand this flip is certainly not an isomorphism in a neighborhood of $S$ and so the MMP terminates by [4.2]. Note that even though the strict transform of $A$ is not ample, in view of (3.9.4) we can replace $\Delta$ by an $\mathbb{R}$-linearly equivalent divisor, such that the hypothesis of [4.2] are satisfied □
5. Log terminal models

Lemma 5.1. Let \( \pi : X \to U \) be a morphism of normal projective varieties. Suppose that \((X, \Delta)\) is a kawamata log terminal pair such that \(\Delta\) is big over \(U\) and \(K_X + \Delta \sim_{\mathbb{R},U} D \geq 0\), for some \(\mathbb{R}\)-divisor \(D\). Suppose that \(K_X + \Theta\) is kawamata log terminal, \(\Theta - \Delta\) is effective and \(\Theta - \Delta\) is supported on the stable fixed divisor of \(K_X + \Delta\). Let \(\phi : X \to Y\) be a birational map of \(\mathbb{Q}\)-factorial varieties over \(U\), whose inverse does not contract any divisors. Suppose that \(K_Y + \Gamma\) is nef over \(U\) and kawamata log terminal, where \(\Gamma = \phi_* \Theta\).

If \(\phi\) only contracts components of the stable fixed divisor of \(K_X + \Delta\), then \(\phi\) is a log terminal model of \((X, \Delta)\) over \(U\).

Proof. We first show that \(\phi\) is a log terminal model of \(K_X + \Theta\). Let \(p : W \to X\) and \(q : W \to Y\) resolve the indeterminacy of \(\phi\). We may write

\[ p^*(K_X + \Theta) + E = q^*(K_Y + \Gamma) + F, \]

where \(E\) and \(F\) are effective, with no common components, and both \(E\) and \(F\) are exceptional for \(q\).

Let \(G = p_* E\). Negativity of contraction (cf. (3.5.2)) implies that it suffices to prove that \(G = 0\). Suppose not. As \(G\) is \(\phi\)-exceptional, \(G + \Theta - \Delta\) is supported on the real stable fixed divisor of \(K_X + \Delta\). As

\[ p_*(p^*(K_X + \Theta) + E) = K_X + \Theta + p_* E \]
\[ = K_X + \Delta + (\Theta - \Delta) + p_* E, \]

implies that the stable fixed divisor of \(p^*(K_X + \Theta) + E\) contains every \(q\)-exceptional divisor which is not \(p\)-exceptional.

Pick a component \(B\) of \(E\) which is not contracted by \(p\). As \(B\) is a component of \(E\) it cannot be a component of \(F\). By the base point free theorem, cf. (3.4.1), \(q^*(K_Y + \Gamma)\) is semiample over \(U\). But this contradicts the fact that \(B\) is a component of the stable fixed divisor of \(p^*(K_X + \Theta) + E\). Thus \(\phi\) is a log terminal model of \(K_X + \Theta\) over \(U\).

By (3.7.7) the support of the stable fixed divisor of \(K_X + \Delta\) and \(K_X + \Theta\) are the same. Thus \(\phi\) must contract every component of the stable fixed divisor of \(K_X + \Delta\). In particular \(\phi_* \Delta = \phi_* \Theta = \Gamma\). But then, by what we have already proved, \(\phi\) is a log terminal model of \((X, \Delta)\) over \(U\). \(\square\)

Lemma 5.2. Assume Theorem B \(n-1\), Theorem C \(n-1\) and Theorem A \(n\).

Let \(\pi : X \to U\) be a morphism of normal projective varieties, where \(X\) has dimension \(n\). Let \(D\) and \(\Delta = A + B\) be two \(\mathbb{R}\)-divisors with the following properties:
(1) the support of $D$ and $\Delta$ has global normal crossings (in particular $X$ is smooth),
(2) $\lceil \Delta \rceil = 0$,
(3) $A$ is ample over $U$ and $B \geq 0$,
(4) $K_X + \Delta \sim_{R, U} D \geq 0$,
(5) $D = M + F$, where every component of $M \geq 0$ is mobile and every component of $F \geq 0$ is a component of the stable fixed divisor of $D$, and
(6) $\Delta$ and $M$ have no components in common.

Then there is a log terminal model $\phi : X \to Y$ for $(X, \Delta)$ over $U$.

Proof. Let $\Theta_0$ be the divisor which contains the support of $M + F$ with coefficient one and otherwise equals $\Delta$. Let $k$ be the number of distinct coefficients in $M$. The idea is to choose a sequence of divisors

$$\Delta \leq \Theta_k \leq \Theta_{k-1} \leq \cdots \leq \Theta_0,$$

and to successively construct a log terminal model for each $K_X + \Theta_i$ over $U$, so that we are going to apply (4.3) $k+1$ times. At the $i$th step, denote by $N_i$ the part of $\Theta_i$ with support equal to the support of $M$ (to simplify the notation, we use indices only on those divisors whose coefficients vary from step to step; if the coefficients do not change, we will abuse notation and use the same symbol for a divisor and its strict transform). We can break the $k+1$ applications of (4.3) into three parts:

Start (Easy: zeroth step) Reduce to the case $K_X + \Theta_0$ is nef over $U$.
Middle (Hard: next $k-1$ steps) Reduce to the case where $N_{k-1}$ is a scalar multiple of $M$.
End (Medium: last step) Eliminate $M$.

At each step, the support of $\Psi_i = \Theta_i - \Delta \geq 0$ is contained in the support of $D$. Suppose that $E$ is a divisor which is contracted at the $i$th step. Then we will prove that $E$ must be a component of $D$. On the other hand, as every component of $M$ is mobile, no component of $M$ is ever contracted. Thus $E$ is a component of the real stable fixed divisor. By (5.1) we are therefore done as soon as $K_Y + \Gamma$ is nef over $U$, where $\Gamma = \phi_\ast \Delta$.

Note that $K_X + \Theta_0$ is divisorially log terminal. Pick an ample divisor $H$ over $U$, such that $K_X + \Theta_0 + H$ is ample over $U$. Let $S_0 = \lceil \Theta_0 \rceil$, $B_0 = \Theta_0 - S_0 - A$ and $C = H$, so that

$$K_X + \Theta_0 + H = K_X + S_0 + A + B_0 + C.$$
If we put \( D_0 = D + \Theta_0 - \Delta \sim_{\mathbb{R},U} K_X + \Theta_0 \) then the support of \( D_0 \) is contained in \( S_0 \). By [1.3], after running a MMP, we may assume that \( K_X + \Theta_0 \) is nef over \( U \).

Suppose that
\[
M = \sum_{i=1}^{k} m_i M_i \quad \text{so that} \quad N_0 = \sum_{i=1}^{k} M_i.
\]

Possibly reordering and grouping together divisors with the same coefficient, we may assume that
\[
0 < m_1 < m_2 < m_3 < \cdots < m_k.
\]
For every \( 1 \leq i \leq k-1 \), we may write,
\[
N_i = \sum_{j=1}^{k} n_{ij} M_j,
\]
where the coefficients \( n_{ij} \) are given by [5.3]. By (2) of [5.3] and since \( n_{k-1,j} = m_j / m_k \), we have
\[
\lambda M = N_{k-1} \leq N_{k-2} \leq \cdots \leq N_1,
\]
where \( \lambda = 1/m_k \).

Let \( G \) be the maximal divisor contained in both \( \Theta_0 - \Delta \) and in Supp\((F)\). For \( 1 \leq i \leq k-1 \), we let
\[
\Psi_i = N_i + G \geq 0 \quad \text{and} \quad \Theta_i = \Psi_i + \Delta.
\]
Note that Supp\(N_i = \) Supp\(M\) and Supp\(G = \) Supp\(F\).

Then
\[
S_i = \Psi \Theta_{i+1} = M_{i+1} + M_{i+2} + \cdots + M_k + [G].
\]
Suppose that, after \( i \) steps, we have constructed a log terminal model for \((X, \Theta_i)\) over \( U \), for \( i \geq 0 \), so that \( K_X + \Theta_i \) is nef over \( U \). By [3.9.4], possibly replacing \( \Theta_i \) by an \( \mathbb{R} \)-linearly equivalent divisor, we may assume that \( \Theta_i \) contains an ample divisor over \( U \). We will abuse notation and continue to refer to this divisor as \( A \).

Let \( C_i = \Theta_i - \Theta_{i+1} = N_i - N_{i+1} \). Property (2) of [5.3] ensures that \( C_i \geq 0 \). Define \( B_{i+1} = \Theta_i - S_{i+1} - A - C_i \). If \( K_X + \Theta_i \) is nef over \( U \) then, at the next step,
\[
K_X + \Theta_{i+1} = K_X + S_{i+1} + A + B_{i+1},
\]
is nef over \( U \).

In order to achieve this, we apply [4.3] to
\[
K_X + \Theta_i = K_X + \Theta_{i+1} + C_i.
\]
To this end, we must choose an $R$-divisor $D_{i+1} \geq 0$ and a constant $\alpha_i > 0$ so that

$$K_X + \Theta_{i+1} \sim_{R,U} D_{i+1} + \alpha_i C_i,$$

and the support of $D_{i+1}$ is contained in $S_{i+1}$.

Since $\Theta_i = \Theta_{i+1} + C_i$, it is enough to choose $D_{i+1}$ and $\alpha_i$ so that

$$D_{i+1} + (\alpha_i + 1)C_i = M + N_i + F + G \sim_{R,U} K_X + \Theta_i,$$

and the coefficient of $M_j$ in $D_{i+1}$ is zero for $j \leq i+1$, so that the support of $D_{i+1}$ is contained in $S_{i+1}$. Since the coefficient of $M_j$ in $C_i$ is zero for $j > i+1$, it automatically follows that $D_{i+1} \geq 0$. Substituting $C_i = N_i - N_{i+1}$ and equating the coefficients of $M_j$ for $j \leq i+1$, we get

$$\alpha_i n_{ij} - (\alpha_i + 1)n_{i+1,j} = m_j,$$

after a little simplification. By (5.3) we may indeed solve for $\alpha_i$ and $n_{ij}$.

Applying (4.3) $k-1$ times, we may assume that $K_X + \Theta_{k-1}$ is nef over $U$, where

$$\Psi_{k-1} = \Theta_{k-1} - \Delta = N_{k-1} + G = \lambda M + G.$$

Let

$$S_k = \cap G = \cup \Theta_{k-1} - M_k,$$

$C_k = N_{k-1} = \lambda M$ and $B_k = \Theta_{k-1} - S_k - C_k - A$. Then

$$K_X + \Theta_k = K_X + S_k + A + B_k \sim_{R,U} F + G + \alpha_k C_k,$$

where $\alpha_k = \frac{1}{\lambda}$. Thus by (4.3) we may assume that

$$K_X + \Theta_k = K_X + \Delta + G,$$

is nef over $U$, and we are done by (5.1) and (3.9.4).

Lemma 5.3. Suppose that

$$0 < m_1 < m_2 < m_3 < \cdots < m_{l+1},$$

is an increasing sequence of positive real numbers.

Then, for $1 \leq i \leq l$ and $1 \leq j \leq l+1$, the system of equations

(i) \[ \alpha_i n_{ij} - (\alpha_i + 1)n_{i+1,j} = m_j \quad \text{for} \quad 1 \leq j \leq i+1 \leq l \]

(ii) \[ n_{ij} = 1 \quad \text{for} \quad 1 \leq i < j \leq l+1 \]

(iii) \[ n_{ij} = \frac{m_j}{m_{i+1}} \quad \text{for} \quad 1 \leq j \leq l+1, \]

for the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{l-1}$ and the $l \times (l+1)$ real matrix $(n_{ij})$ has a unique solution. Further this solution satisfies

1. $0 < \alpha_i$, for all $1 \leq i \leq l-1$,

2. $0 < n_{i+1,j} < n_{ij}$, for all $1 \leq i \leq l-1$ and $1 \leq j \leq i+1$,
(3) $0 < n_{ij}$, for all $1 \leq i \leq l$, $1 \leq j \leq l + 1$,
(4) $n_{ij} < n_{i,j+1}$, for all $1 \leq j \leq i \leq l$, and
(5) $n_{ij} < 1$, for all $1 \leq j \leq i \leq l$.

Proof. We proceed by decreasing induction on $i$ to determine the values of $\alpha_i$ and $n_{ij}$. When $i = l$, (iii) determines the value of $n_{lj}$. Further these values are consistent with (ii), for $i = l$ and $j = l + 1$ and (3), (4) and (5) hold.

Suppose that we have determined the values of $n_{i',j}$ for $i + 1 \leq i' \leq l$ and $\alpha_{i'}$ for $i + 1 \leq i' < l$. (ii) determines the values of $n_{ij} = 1$, for $j > i$. Note that $n_{i+1,i+1} < 1 = n_{i,i+1}$ so (2) holds in this case. Consider the equation from (i)

$$\alpha_i n_{i,i+1} - (\alpha_i + 1)n_{i+1,i+1} = m_{i+1}.$$ 

Since we know that $n_{i,i+1} = 1$ and we have already determined the value of $n_{i+1,i+1}$, we can use this equation to determine the value of

$$\alpha_i = \frac{n_{i+1,i+1} + m_{i+1}}{1 - n_{i+1,i+1}} > 0,$$

which is (1). Finally we then use this value of $\alpha_i$, the values of $n_{i+1,i}$, and the remaining equations from (i)

$$\alpha_i n_{ij} - (\alpha_i + 1)n_{i+1,j} = m_j \quad \text{for } 1 \leq j \leq i \leq l - 1,$$

to determine the value of

$$n_{ij} = \frac{(\alpha_i + 1)n_{i+1,j} + m_j}{\alpha_i}.$$

(2) is clear and this implies (3). (4) is also clear and this implies (5). $\square$

Lemma 5.4. Theorem $[\mathbb{B}]_{i-1}$, Theorem $[\mathbb{C}]_{i-1}$ and Theorem $[\mathbb{A}]_{i}$ imply Theorem $[\mathbb{B}]_{i}$.

Proof. Let $U_0$ be a compactification of $U$, and let $X_0$ be a compactification of $X$, lying over $U_0$. Let $\Delta_0$ be the closure of $\Delta$. By (3.5.5) we may assume that $(X_0, \Delta_0)$ is log smooth. By (3.2.1) we may assume that $K_{X_0} + \Delta_0 \sim_{\mathbb{R}, U_0} D_0 \geq 0$. Since a log terminal model over $U_0$ restricts to a log terminal model over $U$, replacing $X \to U$ by $X_0 \to U_0$, we may assume that $U$ is projective.

Pick an ample Cartier divisor $H$ on $U$ such that $K_X + \Delta + \pi^* H \sim_{\mathbb{R}} D + \pi^* H \geq 0$. Replacing $\Delta$ by $\Delta + \pi^* H$ and $D$ by $D + \pi^* H$, we may assume that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$. By (1) of (3.6.3) we may assume that $\Delta = A + B$, where $A$ is ample over $U$ and $B$ is effective. By (3.7.4) we may assume that $D = M + F$, where every component of $F$ is a component of the stable fixed divisor and every component of $M$ is mobile. We may assume that every component $B$ of $M$ is general in
Pick a log resolution \( f : Y \rightarrow X \) of the support of \( D \) and \( \Delta \), which resolves the base locus of the components of \( M \). Let \( \Gamma \) be the divisor defined in \((3.5.5)\). Note that every component \( \tilde{B} \) of the strict transform \( \tilde{M} \) of \( M \) is still mobile and (by definition of \( \Gamma \)) every component of the exceptional locus belongs to the stable fixed divisor. Possibly replacing \( \Gamma \) by an \( \mathbb{R} \)-linearly equivalent divisor, we may assume that \( \Gamma \) contains an ample divisor. By \((3.5.5)\), we may replace \( X \) by \( Y \) and the result follows by \((5.2)\). \( \square \)

6. Finiteness of models

**Lemma 6.1.** Assume Theorem \( \overline{B} \). Let \( \pi : X \rightarrow U \) be a projective morphism of normal quasi-projective varieties, where \( X \) has dimension \( n \). Fix \( A \), an ample divisor over \( U \). Let \( \mathcal{C} \subset \mathcal{L}_A \) be a rational polytope such that

- For every \( \Delta \in \mathcal{C} \cap \mathcal{P}_A \), there is a divisor \( D \geq 0 \) with \( K_X + \Delta \sim_{\mathbb{R}, U} D \).

Suppose that there is a divisor \( \Delta_0' \) such that \( K_X + \Delta_0' \) is kawamata log terminal.

Then we can find rational maps \( \phi_i : X \rightarrow Y_i \) over \( U \), 1 \( \leq i \leq k \), such that if \( \Delta \in \mathcal{C} \cap \mathcal{P}_A \) then \( \phi_i \) is a log terminal model of \( K_X + \Delta' \) over \( U \), for some \( i \), for any \( K_X + \Delta' \sim_{\mathbb{R}, U} K_X + \Delta \) which is kawamata log terminal. In particular the set of isomorphism classes

\[ \{ Y \mid Y \text{ is the log canonical model of a pair } (X, \Delta), \text{ where } \Delta \in \mathcal{C} \}, \]

is finite.

**Proof.** The proof proceeds by induction on the dimension of \( V \).

We first prove this result under the stronger assumption that there is a divisor \( \Delta_0 \in \mathcal{C} \) such that \( K_X + \Delta_0 \sim_{\mathbb{R}, U} 0 \). Pick \( \Theta \in \mathcal{C} \). Then there is a divisor \( \Delta \) on the boundary of \( \mathcal{C} \) such that \( \Theta \in [\Delta_0, \Delta] \), so that

\[ \Theta - \Delta_0 = \lambda(\Delta - \Delta_0), \]

for some \( \lambda \in [0, 1] \). Now

\[
K_X + \Theta \sim_{\mathbb{R}, U} (K_X + \Theta) - (K_X + \Delta_0) \\
= \Theta - \Delta_0 \\
= \lambda(\Delta - \Delta_0) \\
= \lambda((K_X + \Delta) - (K_X + \Delta_0)) \\
\sim_{\mathbb{R}, U} \lambda(K_X + \Delta).
\]
Thus if $\Theta \neq \Delta_0$, then $K_X + \Delta$ is $\pi$-pseudo-effective iff $K_X + \Theta$ is $\pi$-pseudo-effective and $\phi: X \rightarrow Y$ is a log terminal model for $K_X + \Delta$ over $U$ iff $\phi$ is a log terminal model for $K_X + \Theta$ over $U$. Since the boundary of $C$ is contained in finitely many affine hyperplanes defined over the rationals, we are done by induction on the dimension of $V$.

Now we prove the general case. By compactness it suffices to prove this result locally about any divisor $\Delta_0 \in C$. By (2) of (3.6.3), we may assume that $K_X + \Delta_0$ is Kawamata log terminal.

If $K_X + \Delta_0$ is not $\pi$-pseudo-effective then there is an open neighbourhood $P_0$ of $\Delta_0$ in $L_A$, such that $K_X + \Delta$ is not $\pi$-pseudo-effective for all $\Delta \in P_0$. In this case there are no log terminal models in a neighbourhood of $\Delta_0$.

Thus we may assume that $\Delta_0 \in P_A$. As we are assuming Theorem B, there is a log terminal model $\phi: X \rightarrow Y$ over $U$ for $K_X + \Delta_0$. In particular $\phi$ is $(K_X + \Delta_0)$-negative. But then there is an open neighbourhood $P_0$ of $\Delta_0$ in $C$ such that for any $\Delta \in P_0$, $K_X + \Delta - (\phi^{-1})_* \phi^*(K_X + \Delta)$ is an effective divisor whose support contains every $\phi$-exceptional divisor.

By (3.5.3), replacing $X$ by $Y$, we may assume that $K_X + \Delta_0$ is $\pi$-nef. Let $X \rightarrow Z$ be the ample model of $K_X + \Delta_0$ over $U$, so that $K_X + \Delta_0 \sim_{R,Z} 0$. By what we have already proved there are finitely many birational maps $\phi_i: X \rightarrow Y_i$ over $Z$, $1 \leq i \leq k$, such that for any $\Delta \in C$, where $K_X + \Delta$ is $\pi$-pseudo-effective over $U$ and hence over $Z$, there is an index $i$ such that $\phi_i$ is a log terminal model of $K_X + \Delta$ over $Z$.

Let $\Gamma_0 = \phi_{i_0}^* \Delta_0$ and $\Gamma = \phi_{i_k}^* \Delta$. Since there are only finitely many models $Y_i$, (3.10.3) implies that we may find $\epsilon > 0$ such that if $||\Delta - \Delta_0|| < \epsilon$ then there is an $1 \leq i \leq k$ such that

$$K_{Y_i} + (1 - \epsilon)\Gamma_0 + \epsilon \Gamma = (1 - \epsilon)(K_{Y_i} + \Gamma_0) + \epsilon(K_{Y_i} + \Gamma),$$

is nef over $U$. In particular $Y_i$ is a log terminal model for $K_X + \Delta$ over $U$.\hfill \square

We will need a variation on (3.5.5):

**Lemma 6.2.** Let $\pi: X \rightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $(X, \Delta)$ be a Kawamata log terminal pair, and let $\xi: X \rightarrow Y$ be a weak log canonical model of $K_X + \Delta$ over $U$. Let $\Gamma = \xi_* \Delta$. Let $f: Z \rightarrow X$ be any log resolution of $(X, \Delta)$ such that $\xi \circ f$ is a morphism and suppose that we write

$$K_Z + \Phi_0 = f^*(K_X + \Delta) + E,$$
where \( \Phi_0 \) and \( E \) are effective, with no common components, \( f_* \Phi_0 = \Delta \) and \( E \) is exceptional. Let \( F \) be any effective divisor whose support is equal to the exceptional locus.

Then we may find \( \eta > 0 \) such that if \( \Phi = \Phi_0 + \eta F \) then

- \( f_* \Phi = \Delta \),
- \( K + \Phi \) is Kawamata log terminal,
- if \( \Delta \) is big over \( U \) then so is \( \Phi \), and
- every log terminal model (respectively weak log canonical model) over \( Y \) of \( K + \Phi \) is a log terminal model (respectively weak log canonical model) over \( U \) of \( K + \Delta \).

**Proof.** By (3.5.5), everything is clear, apart from the fact that if \( \phi: Z \to W \) is a log terminal model (respectively weak log canonical model) over \( Y \) of \( K + \Phi \) then it is a log terminal model (respectively weak log canonical model) over \( U \) of \( K + \Delta \).

Let \( \psi: X \to W \) be the induced birational map and set \( \Psi = \phi_* \Phi \). By assumption there is a morphism \( g: W \to Y \). By (3.5.5) possibly blowing up more, we may assume that \( \phi \) is a morphism. By assumption if we write

\[
K + \Phi = \phi^*(K_W + \Psi) + G,
\]

then \( G \geq 0 \) and the support of \( G \) is the full \( \phi \)-exceptional locus (respectively \( G \) is exceptional). Similarly we may write

\[
f^*(K + \Delta) = \phi^*(g^*(K_Y + \Gamma)) + E',
\]

where \( E' \geq 0 \) is exceptional over \( Y \). Thus

\[
\phi^*(K_W + \Psi) + G = \phi^*(g^*(K_Y + \Gamma)) + E' + E + \eta F.
\]

As every component of \( G + E' + E + \eta F \) is exceptional over \( Y \), by negativity of contraction (cf. (3.5.2)) \( G - E' - E - \eta F \geq 0 \). In particular every \( f \)-exceptional divisor is a component of \( G \), so that \( \phi \) contracts every \( f \)-exceptional divisor and so \( \psi^{-1} \) does not contract any divisors. Then \( g_* \Psi = \Gamma \). It follows that \( K_W + \Psi = g^*(K_Y + \Gamma) \). But then \( \phi \) is a \( K_X + \Delta \) log terminal model (respectively a \( K_X + \Delta \) weak log canonical model) over \( U \) by (3.5.3). \( \Box \)

**Lemma 6.3.**

1. **Theorem B**, implies **Theorem C**, with condition (1).
2. **Theorem B**, and **Theorem D**, imply **Theorem C**.

**Proof.** By (3.6.2), \( \mathcal{L}_A \) is a rational polytope. We may assume that \( \mathcal{C} \subseteq \mathcal{L}_A \) is a simplex. Then by (3.6.3) we may assume that for every \( \Delta \in \mathcal{C} \), \( K_X + \Delta \) is Kawamata log terminal. By (3.5.5) we may assume that \( X \) is smooth. Since the property of a birational map \( \phi: X \to Y \)
being a weak log canonical model of a log canonical pair \((X, \Delta)\) only depends on the linear equivalence class of \(K_X + \Delta\), replacing \(A\) by an \(\mathbb{R}\)-linearly equivalent divisor, we may assume that \(A = A_1 + A_2\), where \(A_1\) is an ample divisor over \(U\) and the components of \(A_2\) generate the relative Néron-Severi group of \(X\) over \(U\).

Let \(\phi: X \dasharrow Y\) be a birational morphism over \(U\), whose inverse does not contract any divisors, and let \((Y, \Gamma = \phi_*\Delta)\) be a weak log canonical model over \(U\) of the pair \((X, \Delta)\), where \(\Delta \in C\).

Let \(p: Z \longrightarrow X\) be a log resolution of \((X, \Delta)\) which resolves the indeterminacy locus of \(\phi\). Let \(\Phi\) be the divisor on \(Z\) whose existence is guaranteed by (6.2). As we are assuming Theorem B \[\square\], there is a log terminal model \(\xi: Z \dasharrow W\) of \(K_Z + \Phi\) over \(Y\). In particular there is a morphism \(f: W \longrightarrow Y\), a birational map \(\psi: X \dasharrow W\), and \(K_W + \Psi\) is nef over \(Y\), where \(\Psi = \xi_*\Phi\). By (6.2), \(K_W + \Psi\) is a log terminal model of \(K_X + \Delta\) over \(U\) and \(K_W + \Psi = f^*(K_Y + \Gamma)\).

Pick \(H\) an ample divisor over \(U\) on \(Y\). As the components of \(\psi_*A_2\) generate the relative Néron-Severi group of \(W\) over \(U\), there is a divisor \(B\) on \(W\) with support contained in the support of \(\psi_*A_2\) such that \(B\) and \(f^*H\) are numerically equivalent over \(U\). Then \(B\) and \(B - (K_W + \Psi)\) are nef over \(Y\), and so by the base point free theorem, cf. (3.4.1), \(B\) is \(\mathbb{R}\)-linearly equivalent to the pullback of an ample divisor over \(U\) on \(Y\).

Replacing \(H\) by \(f_*B\), we may assume that \(B = f^*H\), and we may also assume that

\[
K_Y + \Gamma' = K_Y + \Gamma + H,
\]

is kawamata log terminal and ample over \(U\). Let

\[
\Delta' = \Delta + H',
\]

where \(H'\) is the strict transform of \(B\) on \(X\). Possibly rescaling \(H\) we may assume that \(K_X + \Delta'\) is kawamata log terminal and \(\psi\)-negative. Note that then \(\phi\) is \((K_X + \Delta')\)-nonpositive and so \((Y, \Gamma')\) is a log canonical model of \((X, \Delta')\). Note also that \(A_1 \leq \Delta'\).

Let \(V'\) be the vector space spanned by \(V\) and the components of \(A_2\). Let \(\mathcal{I}_\eta\) denote the set of divisors \(G\) on \(X\) with \(\|G\| \leq \eta\) and whose support is contained in the support of \(A_2\). By compactness of \(\mathcal{C}\) we may find a rational number \(\eta > 0\) such that

\[
\mathcal{C}' = \mathcal{C} \times \mathcal{I}_\eta \subset \mathcal{L}_{A_1} \subset V'_{A_1}.
\]

Moreover, we may choose \(\eta\) so that if for every \(\Delta \in \mathcal{C}\) we have \(K_X + \Delta\) is \(\pi\)-big then for every \(\Delta' \in \mathcal{C}'\) then \(K_X + \Delta'\) is \(\pi\)-big. Thus the result follows by (6.1). \(\square\)
In terms of induction, we will need a version of (6.1) locally around the locus where $K_X + \Delta$ is not kawamata log terminal. To this end we need a version of (4.2) for a convex set of divisors:

**Proposition 6.4.** Assume Theorem B and Theorem C with condition (1). Let $(X, \Delta_0 = S + A + B_0 \in P_A)$ be a log smooth projective purely log terminal pair of dimension $n$. Let $V_0$ be the vector space of Weil divisors on $X$ generated by the components of $B_0$. Fix a general ample divisor $H$ such that $K_X + \Delta_0 + H$ is ample, and let $V$ be the translate by $S + A$, of the vector space spanned by $V_0$ and $H$. Given any polytope $F$ in $V$, the cone $C(F)$ over $F$ (with vertex $\Delta_0$) is the polytope spanned by $F$ and $\Delta_0$.

Pick a constant $\alpha > 0$ such that $F = \{ \Delta_0 + E + H \in V | \|E\| \leq \alpha, E \in V_0 \} \subseteq N_A$ and let $C_0 = C(F)$. If $K_X + \Delta_0$ does not have a log terminal model then there is a countable collection of polytopes $P_i$ and birational maps $\phi_i : X \dasharrow Y_i$ such that

1. $P_0^0 \cap P_j^0 = \emptyset$ for $i \neq j$,
2. $P_i \subset P_{Y_i}$,
3. for any $\Delta = \Delta_0 + E + H \in F$, the $(K_X + \Delta_0)$-MMP with scaling of $E + H$ is given by $X \dasharrow Y_{i_1} \dasharrow Y_{i_2} \dasharrow Y_{i_3} \cdots$ for appropriate indices $i_j$,
4. for all $\epsilon > 0$, the set $\{ i \in I | \exists \Delta = \Delta_0 + t(E + H) \in P_i, t > \epsilon \}$ is finite, and
5. if $C_i$ denotes the cone over $P_i$, then $C_i - \{ \Delta_0 \} = \bigcup \{ P_j | P_j \subset C_i \}$.

**Proof.** By assumption the set $P_0 = C_0 \cap N_A$, does not contain $\Delta_0$. Moreover $P_0$ is a polytope. Let $\phi_0 : X \dasharrow Y_0 = X$ be the identity map. Note that $C_0$ is indeed the cone over $P_0$.

Suppose that we have defined $P_1, P_2, \ldots, P_k$, satisfying (1), (2) and a modified version of (3), where we stop the MMP in (3) when $\Delta$ lies on the boundary of $\bigcup P_i$. Let $F_{i,j}$ be the faces of $P_i$, whose associated cones are of maximal dimension. For each such face, let $F = F_{i,j}$ and let $C_{k+1}$ be the corresponding cone. If $P_i$ is of maximal dimension, we discard...
whenever $C_{k+1} = C_i$, for some $i < k$. Let $R$ be a $(K_X + \Delta_0)$-extremal ray of $Y_i$, which cuts out $F$. Let $Y_i \to Y_{k+1}$ be the corresponding step of the $(K_X + \Delta_0)$-MMP.

Let $\mathcal{P}_{k+1}$ be the nef cone of $Y_{k+1}$ intersected with $C_{k+1}$. As before $\mathcal{P}_{k+1}$ is a polytope that, by assumption, does not contain $\Delta_0$.

Thus, by induction, we may assume that we have constructed a countable set of polytopes $\mathcal{P}_i$ and rational maps $\phi_i: X \to Y_i$ satisfying (1), (2) and the modified version of (3) described above. Note that property (4) follows from Theorem C with condition (1), and (3) and (5) follow from (4), since each $C_i$ contains infinitely many $\mathcal{P}_j$.

Lemma 6.5. Assume Theorem $\mathcal{C}_{h-1}$, Theorem $\mathcal{B}_j$, and Theorem $\mathcal{C}_h$ with condition (1). Let $(X, \Delta_0 = S + A + B_0)$ be a purely log terminal pair, where $X$ is projective of dimension $n$, $A$ is ample, $\Delta_0 = S$ and $B_0 \geq 0$. Suppose that $S$ is not a component of $N_\sigma(K_X + \Delta_0)$.

Let $V_0$ be the span of the components of $B_0$. Then we may find an ample divisor $H$, a positive constant $\alpha$, and a log pair $(W, R)$ with the following properties. Let $V$ be the translate by $S + A$ of the vector space of Weil divisors on $X$ generated by $H$ and the components of $B_0$.

Then for every $B \in V_0$ such that

$$\|B - B_0\| < \alpha t,$$

for some $t \in (0, 1]$, we may find a log terminal model $\phi: X \to Y$ of $(X, \Delta = S + A + B + tH)$ which does not contract $S$ and such that the pairs $(W, R)$ and $(Y, T = \phi_*S)$ have isomorphic neighbourhoods of $R$ and $T$.

Proof. Suppose not. Passing to a log resolution, we may assume that $(X, \Delta_0)$ is log smooth. Using the notation established in (6.4) and possibly relabelling, by assumption there is no $C_i$ such that for any two elements $\Delta_1$ and $\Delta_2$ in $C_i$, the corresponding models have isomorphic neighbourhoods of $T$. Hence we may find a sequence of polytopes $\mathcal{P}_i$, such that the corresponding cones are nested $C_i \subset C_{i-1}$, and moreover the corresponding $Y_i$ are not eventually isomorphic in a neighbourhood of $T$. By compactness of $F$, we may find $\Delta \in F$ such that $(\Delta_0, \Delta) \cap C_i \neq \emptyset$ for every $i$. By (3), the corresponding $(K_X + \Delta_0)$-MMP with scaling of $E + H = \Delta - \Delta_0$ is not eventually an isomorphism in a neighbourhood of $S$ and this contradicts (4.2). $\square$

7. Non-vanishing

We follow the general lines of the proof of the non-vanishing theorem, see for example Chapter 3, §5 of [29]. In particular there are two cases:
Lemma 7.1. Assume Theorem B. Let \((X, \Delta)\) be a projective, log smooth, kawamata log terminal pair of dimension \(n\), such that \(K_X + \Delta\) is pseudo-effective and \(\Delta - A \geq 0\) for an ample \(\mathbb{Q}\)-divisor \(A\). Suppose that for every positive integer \(k\) such that \(kA\) is integral,
\[
h^0(X, \mathcal{O}_X(\lceil mk(K_X + \Delta) \rceil + kA)),
\]
is a bounded function of \(m\).

Then there is an \(\mathbb{R}\)-divisor \(D\) such that \(K_X + \Delta \sim \mathbb{R} D \geq 0\).

Proof. By (3.3.2) it follows that \(K_X + \Delta\) is numerically equivalent to \(N_\sigma(K_X + \Delta)\). Since \(N_\sigma(K_X + \Delta) - (K_X + \Delta)\) is numerically trivial and ampleness is a numerical condition, it follows that
\[
\Delta' = A + N_\sigma(K_X + \Delta) - (K_X + \Delta),
\]
is ample and numerically equivalent to \(A\). Thus
\[
K_X + \Delta' = K_X + A' + (\Delta - A),
\]
is numerically equivalent to \(K_X + \Delta\), where \(A' \sim \mathbb{R} A''\) is general and
\[
K_X + \Delta' \sim \mathbb{R} N_\sigma(K_X + \Delta) \geq 0.
\]
Thus by Theorem B, \(K_X + \Delta'\) has a log terminal model \(\phi: X \dashrightarrow Y\), which is then automatically a log terminal model for \(K_X + \Delta\). Replacing \((X, \Delta)\) by \((Y, \Gamma)\) we may therefore assume that \(K_X + \Delta\) is nef and the result follows by the base point free theorem, cf. (3.4.1). \(\square\)

Lemma 7.2. Let \((X, \Delta)\) be a projective, log smooth, kawamata log terminal pair such that \(\Delta = A + B\), where \(A\) is an ample \(\mathbb{Q}\)-divisor and \(B\) is effective. Suppose that there is a positive integer \(k\) such that \(kA\) is integral and
\[
h^0(X, \mathcal{O}_X(\lceil mk(K_X + \Delta) \rceil + kA)),
\]
is an unbounded function of \(m\).

Then we may find a projective, log smooth, purely log terminal pair \((Y, \Gamma)\) and an ample \(\mathbb{Q}\)-divisor \(C\) on \(Y\), where

- \(Y\) is birational to \(X\),
- \(\Gamma - C \geq 0\), and
- \(T = \lceil \Gamma \rceil\) is an irreducible divisor, which is not a component of \(N_\sigma(K_Y + \Gamma)\).

Moreover the pair \((Y, \Gamma)\) has the property that \(K_X + \Delta \sim \mathbb{R} D \geq 0\) for some \(\mathbb{R}\)-divisor \(D\) iff \(K_Y + \Gamma \sim \mathbb{R} G \geq 0\) for some \(\mathbb{R}\)-divisor \(G\).

Proof. Pick \(m\) large enough so that
\[
h^0(X, \mathcal{O}_X(\lceil mk(K_X + \Delta) \rceil + kA)) > \binom{kn + n}{n}.
\]
By standard arguments, given any point \( x \in X \), we may find an effective divisor which is \( \mathbb{R} \)-linearly equivalent to
\[
\omega mk(K_X + \Delta) + kA,
\]
of multiplicity greater than \( kn \) at \( x \). In particular, we may find an effective \( \mathbb{R} \)-divisor
\[
H \sim \mathbb{R} m(K_X + \Delta) + A,
\]
of multiplicity greater than \( n \) at \( x \). Given \( t \in [0, m] \), consider
\[
(t + 1)(K_X + \Delta) = K_X + \frac{m-t}{m} A + B + t(K_X + \Delta + \frac{1}{m} A) \\
\sim_{\mathbb{R}} K_X + \frac{m-t}{m} A + B + \frac{t}{m} H \\
= K_X + \Delta_t.
\]
Fix \( 0 < \epsilon \ll 1 \), let \( A' = \epsilon / mA \) and \( u = m - \epsilon \). We have:

1. \( K_X + \Delta_0 \) is Kawamata log terminal,
2. \( \Delta_t \geq A' \), for any \( t \in [0, u] \) and
3. the locus of log canonical singularities of \((X, \Delta_u)\) contains a very general point of \( X \).

Let \( \pi : Y \longrightarrow X \) be a common log resolution of \((X, \Delta_t)\). We may write
\[
K_Y + \Psi_t = \pi^*(K_X + \Delta_t) + E_t,
\]
where \( E_t \) and \( \Psi_t \) are effective, with no common components, \( \pi_* \Psi_t = \Delta_t \) and \( E_t \) is exceptional. Pick an effective exceptional divisor \( F \) and a positive integer \( l \) such that \( l(\pi^* A' - F) \) is very ample and let \( lC \) be a very general element of the linear system \( |l(\pi^* A' - F)| \). For any \( t \in [0, u] \), let
\[
\Gamma_t = \Psi_t - \pi^* A' + C + F \sim_{\mathbb{R}} \Psi_t.
\]
After cancelling common components of \( \Gamma_t \) and \( N_\sigma(K_Y + \Gamma_t) \), properties (1-3) above become

1. \( K_Y + \Gamma_0 \) is Kawamata log terminal,
2. \( \Gamma_t \geq C \), for any \( t \in [0, u] \) and
3. the locus of log canonical singularities of \((Y, \Gamma_u)\) is not contained in \( N_\sigma(K_Y + \Gamma_u) \).

Moreover

4. \((Y, \Gamma_t)\) is log smooth, for any \( t \in [0, u] \).

Let
\[
s = \sup \{ t \in [0, u] \mid K_Y + \Gamma_t \text{ is log canonical} \}.
\]
Thus, setting \( \Gamma = \Gamma_s \), we may write
\[
\Gamma = T + C + B',
\]
where \( \Gamma \) is ample and \( B' \) is effective. Possibly perturbing \( \Gamma \), we may assume that \( T \) is irreducible, so that \( K_Y + \Gamma \) is purely log terminal and we may assume that \( C \) is \( \mathbb{Q} \)-Cartier.

We will need the following consequence of Kawamata-Viehweg vanishing:

**Lemma 7.3.** Let \((X, \Delta = S + A + B)\) be a \( \mathbb{Q} \)-factorial projective purely log terminal pair and let \( m > 1 \) be an integer. Suppose that

1. \( S = \lceil \Delta \rceil \) is irreducible,
2. \( m(K_X + \Delta) \) is integral,
3. \( h^0(S, \mathcal{O}_S(m(K_X + \Delta))) > 0 \),
4. \( K_X + G + B \) is kawamata log terminal, where \( G \geq 0 \),
5. \( A \sim \mathbb{Q} (m - 1)tH + G \) for some \( t \),
6. \( K_X + \Delta + tH \) is big and nef.

Then \( h^0(X, \mathcal{O}_X(m(K_X + \Delta))) > 0 \).

**Proof.** Considering the long exact sequence associated to the restriction exact sequence,

\[
0 \rightarrow \mathcal{O}_X(m(K_X + \Delta) - S) \rightarrow \mathcal{O}_X(m(K_X + \Delta)) \rightarrow \mathcal{O}_S(m(K_X + \Delta)) \rightarrow 0,
\]

it suffices to observe that

\[
H^1(X, \mathcal{O}_X(m(K_X + \Delta) - S)) = 0,
\]

by Kawamata-Viehweg vanishing, since

\[
m(K_X + \Delta) - S = (m - 1)(K_X + \Delta) + K_X + A + B \\
\sim \mathbb{Q} (m - 1)(K_X + \Delta + tH)
\]

and \( K_X + \Delta + tH \) is big and nef.

**Lemma 7.4.** Theorem \( [\alpha_{-1}] \), Theorem \( B_{\alpha} \) and Theorem \( C_{\alpha} \), with condition (1), imply Theorem \( D_{\alpha} \).

**Proof.** By (3.2.1), it suffices to prove this result for the generic fibre of \( U \). Thus we may assume that \( U \) is a point, so that \( X \) is a projective variety.

By (3.5.5) we may assume that \((X, \Delta)\) is log smooth. By (3.6.3) we may assume that \( \Delta = A + B \), where \( A \) is ample and \( B \geq 0 \). By (7.1) and (7.2), we may therefore assume that \( \Delta = S + A + B \), where \( A \) is an ample \( \mathbb{Q} \)-divisor, \( B \) is effective and \( \lceil \Delta \rceil = S \) is irreducible and not a component of \( N_\sigma(K_X + \Delta) \).

Let \( H \) be the ample divisor on \( X \) and \( \alpha > 0 \) be the constant whose existence is guaranteed by (6.5). Possibly replacing \( A \) be an \( \mathbb{R} \)-linearly equivalent divisor, we may assume that there is a positive constant \( \epsilon \)

\[
...
\]
such that $A - \epsilon H \geq 0$. Let $V_0$ be the vector space of Weil divisors spanned by the components of $B$ and let $V$ be the translate by $S + A$ of the span of $V_0$ and $H$.

Given $t > 0$ and any $B' \in V_0$, $\|B' - B\| < \alpha t$, let
$$
\Psi = S + A + B' + tH.
$$
Let $\phi: X \rightarrow Y$ be the log terminal model of $K_X + \Psi$, whose existence is guaranteed by (6.5). Let $T$ be the strict transform of $S$, let $\Gamma = \phi_* \Psi$ and define $\Theta$ by adjunction
$$
(K_Y + \Gamma)|_T = K_T + \Theta.
$$
By linearity we may formally extend the assignment $\Psi \mapsto \Theta$ to a rational affine linear map
$$
L: V \rightarrow W,
$$
to the whole of $V$, where $W$ is an appropriate finite dimensional rational affine space of Weil divisors on $T$. In particular, $L(\Delta)$ is big and by (6.5) it follows that $K_T + L(\Delta)$ is nef.

Now by taking $A' = \phi_* A|_T$, there is a rational polytope $C_T \subset N_{A'}$ containing $L(\Delta)$ such that $K_T + \Theta$ is kawamata log terminal for any $\Theta \in C_T$, by (1) of (3.6.3) and (3.10.1). Moreover, we can find a rational polytope $C \subset L_A$ containing $\Delta$ such that $L(C) = C_T$.

We may find a positive integer $k$ such that if $r(K_Y + \Gamma)$ is integral then $rk(K_Y + \Gamma)$ is Cartier in a neighbourhood of $T$. By Kollár’s effective base point free theorem, [27], we may find a positive integer $M'$ such that if $D$ is a nef Cartier divisor on $T$ such that $D - (K_T + \Omega)$ is nef and big, where $K_T + \Omega$ is kawamata log terminal, then $M'D$ is base point free, where $M'$ does not depend on either $D$ or on $\Omega$. Set $M = kM'$. Suppose that $\Theta \in C_T$. Then $\Theta \geq A'$, so that we may write
$$
K_T + \Theta \sim_\mathbb{R} K_T + G + \Theta',
$$
where $G$ is ample and $K_T + G + \Theta'$ is kawamata log terminal. Now if $rk(K_Y + \Gamma)|_T = rk(K_T + \Theta)$, then
$$
k(T + \Theta) - (K_T + \Theta') \sim_\mathbb{R} (rk - 1)(K_T + \Theta) + G.
$$
Thus, if $r(K_Y + \Gamma)$ is integral, then $Mr(K_T + \Theta)$ is base point free.

By (3.6.5), there are real numbers $r_i > 0$ with $\sum r_i = 1$, positive integers $p_i > 0$ and $\mathbb{Q}$-divisors $\Delta_i \in C$ such that
$$
p_i(K_X + \Delta_i),
$$
is integral,
$$
K_X + \Delta = \sum_{59} r_i(K_X + \Delta_i),
$$
and
\[ \| \Delta - \Delta_i \| \leq \frac{\alpha \epsilon}{m_i}, \]
where \( m_i = Mp_i \). Let \( \Theta_i = L(\Delta_i) \).

By our choice of \( k, p_i k(K_T + \Theta_i) \) is Cartier. So, \( m_i(K_T + \Theta_i) \) is base point free and so
\[ h^0(T, \mathcal{O}_T(m_i(K_T + \Theta_i))) > 0. \]
(7.3) implies that \( h^0(Y, \mathcal{O}_Y(m_i(K_Y + \Gamma_i))) > 0 \), where \( \Gamma_i = \phi_* \Delta_i \).

Notice that the pair \((Y, \Gamma_i = T + \phi_* A + \phi_* B)\) clearly satisfies conditions (1-4) of (7.3). We then let \( t = \epsilon/m_i \) so that \( \phi_* A \geq (m - 1)t \phi_* H \) and condition (6) of (7.3) holds. Conditions (5) and (7) of (7.3) are now easy to check.

As \( \phi \) is \((K_X + \Delta_i + tH)\)-negative, it is certainly \((K_X + \Delta_i)\)-negative. But then
\[ h^0(X, \mathcal{O}_X(m_i(K_X + \Delta_i))) = h^0(Y, \mathcal{O}_Y(m_i(K_Y + \Gamma_i))) > 0. \]
In particular there is an \( \mathbb{R} \)-divisor \( D \) such that
\[ K_X + \Delta = \sum r_i(K_X + \Delta_i) \sim_{\mathbb{R}} D \geq 0. \]

8. Proof of Theorems

Proof of Theorem B, Theorem C and Theorem D. Immediate from the main result of [12], (5.4), (6.3) and (7.4).

Proof of (1.2). By (3.5.5) we may assume that \( \pi \) is projective and \((X, \Delta)\) is log smooth. Suppose \( K_X + \Delta \) is big. Then we may write \( K_X + \Delta \sim_{\mathbb{R}, U} B \geq 0 \). As \( K_X + \Delta + \epsilon B \) is kawamata log terminal for \( \epsilon \) sufficiently small, and a log terminal model for \( K_X + \Delta + \epsilon B \) over \( U \) is the same as a log terminal model for \( K_X + \Delta \) over \( U \), replacing \( \Delta \) by \( \Delta + \epsilon B \) we may assume that \( \Delta \) is big over \( U \). (1) then follows by Theorem B and Theorem D.

(2) and (3) follow from (1) and (3.4.2).

9. Proof of Corollaries

Proof of (1.1.1). (1), (2) and (3) are immediate from (1.2). (4) is Theorem D of [5].

Proof of (1.1.2). Immediate by Theorem (5.2) of [8] and (3) of (1.2).
Proof of (1.1.3). Replacing \((Y_i, \Gamma_i)\) by terminal models, we may assume that \(K_{Y_i} + \Gamma_i\) is terminal. In this case it is well known that \(Y_1\) and \(Y_2\) are isomorphic in codimension one (cf. [13] and [21]).

Replacing \(U\) by the common ample model of \((X, \Delta)\), we may assume that \(K_{Y_i} + \Gamma_i\) is numerically trivial over \(U\). Let \(H_1\) be the strict transform on \(Y_2\). Possibly replacing \(H_2\) by a small multiple, we may assume that \(K_{Y_i} + \Gamma_i + H_i\) is kawamata log terminal.

Suppose that \((K_{Y_1} + \Gamma_1 + H_1)\) is not nef over \(U\). Then there is a \((K_{Y_1} + \Gamma_1 + H_1)\)-flip over \(U\) which is automatically a \((K_{Y_1} + \Gamma_1)\)-flop over \(U\). By finiteness of log terminal models for \(K_X + \Delta\), this \((K_{Y_1} + \Gamma_1 + H_1)\)-MMP terminates. Thus we may assume that \(K_{Y_1} + \Gamma_1 + H_1\) is nef over \(U\). But then \(K_{Y_2} + \Gamma_2 + H_2\) is the corresponding log canonical model, and so there is a small birational morphism \(f: Y_1 \to Y_2\). As \(Y_2\) is \(Q\)-factorial, \(f\) is an isomorphism. \(\square\)

Proof of (1.1.5). We first prove (1) and (2). By Theorem C and (3.10.2), and since log canonical models are unique, it suffices to prove that if \(\Delta \in \mathcal{P}_A\) then \(K_X + \Delta\) has a weak log canonical model. By (2) of (3.6.3) we may assume that \(K_X + \Delta\) is kawamata log terminal, and we may apply (1.2). Thus we may assume that \(K_X + \Delta\) is nef over \(U\) and the result follows from the base point free theorem, cf. (3.4.1).

(3) follows as in the proof of (3.10.2). \(\square\)

Proof of (1.1.7). Immediate consequence of (1.1.5). \(\square\)

Proof of (1.1.9). Let \(\psi_j: X \to Z_j, 1 \leq j \leq q\) be the finitely many ample models whose existence is guaranteed by (1.1.5). Let \(\mathcal{C}\) be the polytope spanned by \(\Delta_1, \Delta_2, \ldots, \Delta_k\) and let \(\mathcal{C}_j\) be the subpolytope given by the closure of those divisors with ample model \(\psi_j\).

Replacing \(\Delta_1, \Delta_2, \ldots, \Delta_k\) by the extreme points of \(\mathcal{C}_j\), we may assume that \(\mathcal{C} = \mathcal{C}_j\) and so we may drop the index \(j\). Pick \(n_i\) such that \(D_i = n_i(K_X + \Delta_i)\) is integral. Passing to a truncation it suffices to prove that

\[
\mathcal{R}(\pi, D^\bullet) = \bigoplus_{m \in \mathbb{N}^k} \pi_* O_X(\sum m_i D_i),
\]

is a finitely generated \(O_U\)-module. Passing to a truncation, we may assume that

\[
D_i = \psi^* H_i,
\]

where \(H_i\) is semiample on \(Z\). In this case

\[
\mathcal{R}(\pi, D^\bullet) = \mathcal{R}(\pi', H^\bullet),
\]

where \(\pi': X' \to Z\) is obtained from \(\pi\) by replacing each \(\psi_j\) by \(\psi'\). \(\square\)
where \( \pi' : Z \to U \) is the corresponding morphism. But the latter Cox ring is finitely generated since the \( \psi^* H_i \) are divisors on \( Z \) which are semiample over \( U \).

\[ \square \]

**Proof of (1.2.2).** We follow the argument given in [17]. Fix a positive real number \( d \). Consider the set of pairs \( (X, \Delta) \) such that

(i) \( X \) has dimension \( n \),
(ii) \( K_X + \Delta \) is kawamata log terminal,
(iii) \( K_X + \Delta \) is ample,
(iv) \( r(K_X + \Delta) \) is Cartier, and
(v) the degree \( (K_X + \Delta)^n \) < \( d \).

By the effective base point theorem of Kollár, see [27], there is a positive integer \( M \) such that if \( (X, \Delta) \) is any pair satisfying (i-iv) then \( M(K_X + \Delta) \) is very ample. Thus for the family of all pairs \( (X, \Delta) \) satisfying (i-v), the pairs \( (X, \mathcal{O}_X(M(K_X + \Delta))) \) belong to a bounded family. It is then easy to see that the pairs \( (X, \Delta) \) themselves also belong to a bounded family.

Thus we may find a pair \( (Y, \Gamma) \) and a projective family \( \pi : Y \to U \), over a quasi-projective base, such that every pair \( (X, \Delta) \) is isomorphic to a fibre of this family, and any fibre over an open subset is a pair satisfying (i-v). Replacing \( U \) and \( Y \) by compactifications, we may assume that \( U \) is projective. Note that any pair \( (X, \Delta) \) satisfying (i-iv) is its own log canonical model.

By Theorem 2.1 of [1], there is a proper morphism \( U' \to U \), where \( U' \) is smooth and a birational map \( Y' \to Y \times_U U' \) and \( K_{Y'} + \Gamma' \) is kawamata log terminal and every fibre is semi log canonical, where \( \Gamma' \) is the sum of the divisors of log discrepancy less than one which dominate \( U' \). By Theorem 0.3 [1], after a further base change we may assume in addition that \( Y' \to U' \) has reduced fibres. By [1,2] the relative log canonical model \( Y'' \to Y'' \) over \( U' \) exists. Note that the fibres of \( Y \to U \) satisfying (i-iv) are isomorphic to the fibres of \( Y'' \to U' \), as the log canonical model is unique. Replacing \( Y \to U \) by \( Y'' \to U' \) we may therefore assume that \( (Y, \Gamma) \) is kawamata log terminal. Note that every fibre is smoothable of type \( (h,r) \).

Pick a positive integer \( m \) such that \( m(K_Y + \Gamma) \) is Cartier. Possibly replacing \( m \) by a multiple, we may assume \( m(K_{X_t} + \Delta_t) \) is very ample, for every fibre of \( \pi \). Thus we may assume that \( X_t \) is embedded in \( \mathbb{P}^N \) by the linear system \( |m(K_{X_t} + \Delta_t)| \), where \( N = h(m) - 1 \) is independent of \( t \). The locus \( H \) corresponding to pairs \( (X, \Delta) \) which are embedded in \( \mathbb{P}^N \) by \( |m(K_X + \Delta)| \) is a locally closed subset of the Hilbert scheme.
Suppose that \((X, \Delta)\) is a smoothable pair of index \(r\). By assumption there is a pair \((W, \Phi)\), and flat family \(\psi: W \to Z\) over a smooth curve \(Z\), such that one reduced fibre \((W_0, \Phi_0)\) is isomorphic to \((X, \Delta)\) and the general reduced fibre \((W_t, \Phi_t)\) is kawamata log terminal and \(r(K_{W_t} + \Phi_t)\) is Cartier. By (1.4.4) \(K_{W_t} + \Phi\) is kawamata log terminal. Now by the universal property of the Hilbert scheme, we may find a map of \(Z \setminus \{0\}\) to \(H\). Possibly applying a family of projective automorphisms, we may assume that the image lies in \(U \subset H\). As \(U\) is projective, we may complete this to a map \(Z \to Y\), and pullback to get another family \(\psi': W' \to Z\) over \(Z\). These families must be isomorphic, by uniqueness of log canonical models, so that every smoothable pair of type \((h, r)\) occurs in \(H\).

Let \(G = \text{PGL}(N + 1)\). Since every point of \(H\) corresponds to a semi log canonical pair \((X, \Delta)\) such that \(K_X + \Delta\) is ample, the set of elements of \(G\) which stabilise the point \([X] \in H\) is finite. Thus we may form quotient stack \(\mathcal{M} = [H/G]\). Note that there is a surjective morphism \(U \to \mathcal{M}\). By the main result of [24] there is a coarse moduli space \(\mathcal{M} \to M\), where \(M\) is an algebraic space. Note that if \(\Delta\) is empty, then both \(\mathcal{M}\) and \(M\) are separated.

Fix a positive integer \(k\) divisible by \(m\). Given a family \(\psi: W \to B\) of smoothable varieties of type \((h, r)\) consider the vector bundle \(\psi^* (O_W(k(K_W + \Phi)))\). Then there is a corresponding vector bundle \(V_k\) on \(\mathcal{M}\). Let \(L_k\) be the determinant of \(V_k\). Then some tensor power \(p = p_k\) of \(L_k\) descends to a line bundle \(L_k\) on \(M\).

Let \(Z \subset M\) be a proper subspace. Let \(Z = Z \times_M \mathcal{M}\) be the fibre product. Let \(Z'\) be the coarse moduli space of \(Z\). Then the natural morphism \(Z' \to Z\) is finite. In particular \(Z'\) is proper and \(Z'\) is projective iff \(Z\) is projective. Replacing \(M\) and \(\mathcal{M}\) by \(Z'\) and \(Z\) we may assume that \(M\) is proper and it suffices to prove that \(L_k\) is ample. By a straightforward modification of the proof of (2.6) of [44], we may find a finite surjective morphism \(X \to \mathcal{M}\), where \(X\) is a proper scheme. By the main result of [25] the line bundles \(L_k\) are ample, since their pullback to \(X\) is ample and there is a family \(p: Y \to X\) over \(X\).

**Proof of (1.2.3).** The set of polarised pairs \((X, M)\), where \(X\) is kawamata log terminal, and \(M\) is ample, with fixed hilbert polynomial \(h\) is bounded. It follows that we may find a positive integers \(p\) and \(r\) such that \(L(-rK_X)\) is always very ample, where \(L = M^\otimes p\). Let \(r\Delta\) be a general element of the linear system \(|L(-rK_X)|\). Then \(L = \mathcal{O}_X(r(K_X + \Delta))\) and \(K_X + \Delta\) is kawamata log terminal.

The result now follows by (1.2.2).
We will need a well-known result on the geometry of the moduli spaces of \( n \)-pointed curves of genus \( g \):

**Lemma 9.1.** Let \( X = \overline{M}_{g,n} \) and let \( D \) be the sum of the boundary divisors.

Then

- \( X \) is \( \mathbb{Q} \)-factorial,
- \( K_X \) is kawamata log terminal and
- \( K_X + D \) is log canonical and ample.

**Proof.** \( X \) is \( \mathbb{Q} \)-factorial, \( K_X \) is kawamata log terminal and \( K_X + D \) is log canonical as the pair \((X, D)\) is locally a quotient of a normal crossings pair.

It is proved in [33] that \( K_X + D \) is ample when \( n = 0 \). To prove the general case, consider the natural map

\[ \pi: \overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}, \]

which drops the last point. Let \( Y = \overline{M}_{g,n+1} \) and \( G \) be the sum of the boundary divisors. Then \( K_Y + G \) is ample on the fibres of \( \pi \) by adjunction and the definition of a stable pair. On the other hand, we may write

\[ K_Y + G = \pi^*(K_X + D) + \psi, \]

for some \( \mathbb{Q} \)-divisor \( \psi \). It is proved in [9] that \( \psi \) is nef and so the result follows by induction on \( n \). \( \square \)

**Proof of (1.2.4).** By (9.1) \( K_X + D \) is ample and log canonical, where \( D \) is the sum of the boundary divisors. In particular \( K_X + \Delta \) is kawamata log terminal, provided none of the \( a_i \) are equal to one.

Pick \( k \) such that \( k(K_X + D) \) is very ample, and pick a general element \( kA' \in |k(K_X + D)| \). Let \( A = \delta A' \). Note that

\[ (1 + \delta)(K_X + \Delta) = K_X + \delta(K_X + D) + (1 + \delta)\Delta - \delta D \]

\[ \sim_{\mathbb{R}} K_X + A + B. \]

Now

\[ 0 \leq (\Delta - \delta D) + \delta \Delta = B = \Delta + \delta(\Delta - D) \leq D. \]

Thus the result is an immediate consequence of Theorem [B], Theorem [C] and Theorem [D]. \( \square \)

**Proof of (1.3.1).** Immediate from (1.1.9) and the main result of [16]. \( \square \)

**Proof of (1.3.2).** Pick any \( \pi \)-ample divisor \( A \) such that \( K_X + \Delta + A \) is \( \pi \)-ample and kawamata log terminal. We may find \( \epsilon > 0 \) such that \( K_X + \Delta + \epsilon A \) is not \( \pi \)-pseudo-effective. We run the \((K_X + \Delta + \epsilon A)\)-MMP

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over $U$ with scaling of $A$. Since every step of this MMP is $A$-positive, it is automatically a $(K_X + \Delta)$-MMP as well. But as $K_X + \Delta + \epsilon A$ is not $\pi$-pseudo-effective, this must terminate with a Mori fibre space $g: Y \rightarrow W$ over $U$.

**Proof of (1.3.4).** As $X$ is a Mori dream space, the cone of pseudo-effective divisors is a rational polyhedron. It follows that $\overline{\text{NE}}(X)$ is also a rational polyhedron, as it is the dual of the cone of pseudo-effective divisors.

The description of an extremal ray $F$ of $\overline{\text{NE}}(X)$ then follows by taking a kawamata log terminal divisor $K_X + \Theta$ which supports $F$ and applying (1.3.2).

**Proof of (1.4.1).** The flip of $\pi$ is precisely the log canonical model, so that this result follows from Theorem B.

**Proof of (1.4.2).** It suffices to prove that we cannot find an infinite sequence of flips for the MMP with scaling, and this follows from (4.1).

**Proof of (1.4.3).** Perturbing $\Delta$ by an ample divisor which contains the centre of every element of $\mathcal{E}$ of log discrepancy one, we may assume that $\mathcal{E}$ contains no exceptional divisors of log discrepancy one. Replacing $\Delta$ by $(1 - \epsilon)\Delta + \epsilon \Delta_0$ we may assume that $K_X + \Delta$ is kawamata log terminal. We may write

$$K_Y + \Psi = f^*(K_X + \Delta) + E,$$

where $\Psi$ and $E$ are effective divisors, with no common components, $f_*\Psi = \Delta$ and $E$ is exceptional. Let $F$ be the reduced sum of all the exceptional divisors which are not components of $E$ nor elements of $\mathcal{E}$.

Pick $\epsilon > 0$ such that $K_Y + \Gamma = K_Y + \Psi + \epsilon F$ is kawamata log terminal. As $f$ is birational $\Gamma$ is big over $X$ and so by (1.2) we may find a log terminal model $\phi: Y \rightarrow W$ over $X$ for $(Y, \Gamma)$.

Replacing $Y$ by $W$ we may assume that $K_Y + \Gamma$ is nef over $U$. By negativity of contraction it follows that $E + \epsilon F$ is zero, so that

$$K_Y + \Psi = f^*(K_X + \Delta).$$

But then we must have contracted every exceptional divisor which does not belong to $\mathcal{E}$.

**Proof of (1.4.4).** Since $(X, \Delta)$ is purely log terminal near the image of $S$ if and only if $(S, \Theta)$ is kawamata log terminal by (17.6) of [28], and since $(X, \Delta)$ is log canonical near the image of $S$ if and only if $(S, \Theta)$ is log canonical (cf. [18]), we may assume that $(X, \Delta)$ is purely log terminal near the image of $S$ and $(S, \Theta)$ is kawamata log terminal. At
this point we simply follow the proof of (17.12) of [28], substituting (1.4.3) for (17.10) of [28]. □

Proof of (1.4.5). Since this result is local in the étale topology, we may assume that $U$ is affine. Let $f : Y \rightarrow X$ be a log resolution of $(X, \Delta)$, so that the composition $\psi : Y \rightarrow U$ of $f$ and $\pi$ is projective. We may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma$ and $E$ are effective with no common components and $E$ is $f$-exceptional.

Pick a $\psi$-ample divisor $H$ such that $K_Y + \Gamma + H$ is $\psi$-ample. We run the $(K_Y + \Gamma)$-MMP over $U$ with scaling of $H$. Since $X$ contains no rational curves contracted by $\pi$, this MMP is automatically a MMP over $X$. Since $\Gamma$ is big over $X$ and termination is local in the étale topology, this MMP terminates by (1.4.2).

Thus we may assume that $K_Y + \Gamma$ is $f$-nef, so that $E$ is empty, and $f$ is small. As $X$ is analytically $\mathbb{Q}$-factorial it follows that $f$ is an isomorphism. But then $\pi$ is a log terminal model. □

References


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