Linear systems and Fano varieties: singularities of linear systems

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References:
[B-1] Anti-pluricanonical systems on Fano varieties.
[B-2] Singularities of linear systems and boundedness of Fano varieties.

Singularities: lc thresholds

Let (X, B) be an lc pair and $L \geq 0$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor.

Define the **lc threshold**

$$\mathrm{lct}(X, B, L) = \sup\{s \mid (X, B + sL) \text{ is lc}\}$$

which measures singularities of L with respect to (X, B).

Example: X a curve. Then

$$\mathrm{lct}(X, B, L) = \min\{\frac{1 - \mu_x B}{\mu_x L}\}_{x \in X}.$$

Example: $(X, B + L)$ log smooth. Then

$$\mathrm{lct}(X, B, L) = \min\{\frac{1 - \mu_D B}{\mu_D L}\}_{D \subset X \text{ prime divisor}}.$$

Example: $X = \mathbb{P}^2$, $B = 0$, L a nodal curve. Then

$$\mathrm{lct}(X, B, L) = 1.$$

Singularities: lc thresholds of linear systems

Let (X, B) be a pair and A be an $\mathbb{R}$-divisor.

Let $|A|_{\mathbb{R}} = \{N \mid 0 \leq N \sim_{\mathbb{R}} A\}$.

Define the **lc threshold**

$$\operatorname{lct}(X, B, |A|_{\mathbb{R}}) = \sup\{s \mid (X, B + sN) \text{ is lc for every } 0 \leq N \sim_{\mathbb{R}} A\}$$

Example: $X = \mathbb{P}^d$, $B = 0$, and $A = -K_X$, then

$$\operatorname{lct}(X, B, |A|_{\mathbb{R}}) = \frac{1}{d+1}.$$

Example: $X \subset \mathbb{P}^d$ is a smooth hypersurface of degree $r \leq d$, $B = 0$, and $A = -K_X$, then

$$\operatorname{lct}(X, B, |A|_{\mathbb{R}}) = \frac{1}{d+1-r}$$

[Cheltsov-Shramov].

Singularities: lc thresholds of linear systems

The threshold is especially important when X is projective and A is ample. Because of connections with *stability*.

In that context, the lc threshold is also called *global lc threshold* or α*-invariant*.

But we are interested in it for different reasons. It plays a crucial role in the proof of BAB, etc.

Theorem ([B-2])

For each $d, r \in \mathbb{N}$ *and* $\epsilon \in \mathbb{R}^{>0}$ *there is* $t \in \mathbb{R}^{>0}$ *such that if*

- (X, B) *is projective* ϵ*-lc of dimension* d*,*
- A *is very ample with* $A^d \leq r$*, and*
- $A - B$ *is ample,*

then

$$\mathrm{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

Singularities: lc thresholds of linear systems

Theorem ([B-2])

For each $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$ there is $t \in \mathbb{R}^{>0}$ such that if

- *(X, B) is a projective ϵ-lc pair of dimension d, and*
- *$A := -(K_X + B)$ is nef and big,*

then

$$\mathrm{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

This was conjectured by Ambro who proved it in the toric case.

It can be derived from BAB but in reality it is used in the proof of BAB.

On the other hand, one can ask whether the lc threshold is a rational number.

More precisely, Tian asked if the lc threshold is calculated by some element of $|A|_{\mathbb{R}}$, for Fano's.

Singularities: lc thresholds of linear systems

Theorem ([B-2])

Let (X, B) be a projective klt pair such that $A := -(K_X + B)$ is nef and big. Assume that $\operatorname{lct}(X, B, |A|_{\mathbb{R}}) \leq 1$.

Then there is $0 \leq D \sim_{\mathbb{R}} A$ such that

$$\operatorname{lct}(X, B, |A|_{\mathbb{R}}) = \operatorname{lct}(X, B, D).$$

Moreover, if B is a $\mathbb{Q}$-divisor, then we can choose $D \sim_{\mathbb{Q}} A$, hence in particular, the lc threshold is a rational number.

Shokurov has an unpublished proof of the theorem in dimension two.

Boundedness of lc thresholds: proof

Lets recall the main result.

Theorem ([B-2])

For each $d, r \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$ there is $t \in \mathbb{R}^{>0}$ such that if

- *(X, B) is projective ϵ-lc of dimension d,*
- *A is very ample with $A^d \leq r$, and*
- *$A - B$ is ample,*

then

$$\operatorname{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

Step one: set up.

Pick $0 \leq N \sim_{\mathbb{R}} A$.

Let s be the largest number such that $(X, B + sN)$ is ϵ'-lc where $\epsilon' = \frac{\epsilon}{2}$.

Boundedness of lc thresholds: proof

Enough to show s is bounded from below.

There is a prime divisor T on birational models of X with log discrepancy

$$a(T, X, \Delta) = \epsilon'$$

where $\Delta := B + sN$.

Enough to show multiplicity of T in $\phi^* N$ is bounded on some resolution $\phi \colon V \to X$ on which T is a divisor.

We can assume the image of T on X is a closed point x otherwise we can cut by hyperplane sections and apply induction on dimension.

There is a birational contraction $Y \to X$ contracting exactly T.

Step two: toroidalisation.

A key ingredient here is provided by the theory of complements.

Boundedness of lc thresholds: proof

Using ampleness of $-(K_Y + T)$ over X, we can find Λ_Y such that (Y, Λ_Y) is lc near T and $n(K_Y + \Lambda_Y) \sim 0/X$ for a bounded $n \in \mathbb{N}$.

We can think of $K_Y + \Lambda_Y$ as a local-global type of complement.

Crucial point: if Λ is the pushdown of Λ_Y, then we can assume $A - \Lambda$ is ample.

By construction, the log discrepancy $a(T, X, \Lambda) = 0$ and $(X, \operatorname{Supp} \Lambda)$ is bounded.

Using resolution of singularities we can assume (X, Λ) is log smooth and Λ is reduced.

The advantage of having Λ is: now T can be obtained by a sequence of blowups, toroidal with respect to (X, Λ).

Boundedness of lc thresholds: proof

The first step is just the blowup of x.

One argues that it is enough to bound the number of these blowups.

We can discard any component of Λ not passing through x, say $\Lambda = S_1 + \cdots + S_d$.

A careful analysis of $Y \to X$ allows us to modify the situation so that $\operatorname{Supp} \Delta$ does not contain any stratum of (X, Λ) apart from x.

This is one of the difficult steps of the whole proof.

Step three: torification.

Since (X, Λ) is log smooth and bounded,

we can find a surjective finite morphism $X \to \mathbb{P}^d$ such that

it maps x to the origin $z = (0 : \cdots : 0 : 1)$,

Boundedness of lc thresholds: proof

and it maps S_i on H_i where $H_1, \dots, H_d$ are the coordinate hyperplanes passing through z.

Since $\operatorname{Supp} \Delta$ does not contain any stratum of (X, Λ) apart from x, it is not hard to reduce the problem to a similar problem on $\mathbb{P}^d$.

From now on we assume $X = \mathbb{P}^d$ and that S_i are the coordinate hyperplanes.

The point of this reduction is that now (X, Λ) is not only toroidal but actually toric,

and $-(K_X + \Lambda)$ is very ample.

In particular, we can modify Δ so that $K_X + \Delta$ is numerically trivial.

Let $W \to X$ be the sequence of blowups which obtains T.

Boundedness of lc thresholds: proof

Since the blowups are toric, W is a toric variety.

If $Y \to X$ is the birational morphism contracting T only, as before, then Y is also a toric variety.

Moreover, if $K_Y + \Delta_Y$ is the pullback of $K_X + \Delta$, then (Y, Δ_Y) is ϵ'-lc and $K_Y + \Delta_Y$ is numerically trivial.

Running MMP on $-K_Y$ another toric variety Y' which is (weak) Fano and ϵ'-lc.

By the toric version of BAB, Y' belongs to a bounded family.

From this we can produce a klt strong m-complement $K_{Y'} + \Omega_{Y'}$ for some bounded $m \in \mathbb{N}$

which induces a klt strong m-complement $K_Y + \Omega_Y$ which in turn gives a klt strong m-complement $K_X + \Omega$.

Boundedness of lc thresholds: proof

Also Ω belongs to a bounded family as its coefficients are in a fixed finite set and its degree is bounded.

This implies that $(X, \Omega + u\Lambda)$ is klt for some $u > 0$ bounded from below.

An easy calculation bounds the multiplicity of T in the pullback of Λ on W is bounded from above

which in turn implies the number of blowups in $W \to X$ is bounded as required.

Example

Assume $(X = \mathbb{P}^2, B)$ is ϵ-lc and $S \subset X$ is a line.

Let $L = A = lS$ where $l \in \mathbb{N}$.

Multiplicity of L at any point $x \in L$ is l,

so $\mathrm{lct}(X, B, L) \leq \frac{1}{l}$.

So the larger is l, the smaller is the threshold.

Next we illustrate how the threshold depends on the degree of B by constructing certain B.

Let T be another line and $\{x\} = S \cap T$.

Let $X_1 \to X$ be the blowup at x.

Let x_1 be the intersection of the exceptional divisor E_1 and $S^\sim$.

Example

Let $X_2 \to X_1$ be the blowup at x_1.

Let x_2 be the intersection of the new exceptional divisor E_2 and $S^\sim$.

At each step we blowup the intersection point of $S^\sim$ and the newest exceptional divisor.

Put $W := X_r$.

The exceptional locus of $\phi\colon W \to X$ consists of a chain of curves all of which are -2-curves except one which is a -1-curve.

Then $-K_W$ is nef over X.

In fact, $-aK_W$ is base point free over X, for some $a \in \mathbb{N}$.

Thus there is $0 \leq B_W \sim_{\mathbb{R}} \alpha\phi^* H - K_W$ for some $\alpha > 0$ such that (W, B_W) is $\frac{1}{2}$-lc and $K_W + B_W \sim_{\mathbb{R}} 0/X$.

Example

Let B be the pushdown of B_W.

Then (X, B) is $\frac{1}{2}$-lc.

Now let $L = S + T$.

Then the coefficient of E_r in $\phi^* L$ is $r + 1$.

So

$$\operatorname{lct}(X, B, L) = \operatorname{lct}(W, B_W, \phi^* L) \leq \frac{1}{r+1}.$$

Thus there is no lower bound on the lc threshold if r is arbitrarily large.

This does not contradict the theorem on boundedness of lc thresholds because when $r \gg 0$, the divisor $A - B$ cannot be ample (here $A = lS$ with l fixed).