Linear systems and Fano varieties: complements and effective birationality

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References:
[B-1] Anti-pluricanonical systems on Fano varieties.
[B-2] Singularities of linear systems and boundedness of Fano varieties.

Singularities of pairs

A **pair** (X, B) consists of a normal variety X and a boundary divisor B with coefficients in $[0, 1]$.

Singularities of (X, B) are defined by taking a log resolution $\phi\colon W \to X$ and writing $K_W + B_W = \phi^*(K_X + B)$.

The larger the coefficients of B_W, the worse the singularities.

Singularities are lc (resp. klt) if every coefficient of B_W is ≤ 1 (resp. < 1).

(X, B) is ϵ-lc if every coefficient of B_W is $\leq 1 - \epsilon$.

The **log discrepancy** of a prime divisor D is defined as

$$a(D, X, B) = 1 - \mu_D B_W.$$

A **non-klt centre** is the image on X of a component of B_W with coefficient ≥ 1.

Fano pairs

Let (X, G) be a klt pair, and $f\colon X \to Z$ be a contraction.

We say (X, G) is **Fano over** Z if $-(K_X + G)$ is ample over Z.

We say X is **Fano type over** Z.

We can run MMP on any $\mathbb{R}$-divisor on X, over Z.

Running MMP on $-K_X$ over Z ends with Y such that Y is weak Fano over Z, i.e. $-K_Y$ is nef and big over Z.

Example: $X \to Z$ toric morphism, then X is Fano type over Z.

Complements

Let (X, B) be a pair and $X \to Z$ a contraction.

Let $z \in Z$.

A **strong n-complement** of $K_X + B$ over z is of the form $K_X + B^+$ such that over z we have

$$\begin{cases} (X, B^+) \text{ has lc singularities,} \\ n(K_X + B^+) \sim 0 \\ B^+ \geq B \end{cases}$$

Note that $n(B^+ - B) \in |-n(K_X + B)|_{/z}$.

Such complements do not always exist, e.g. $K_X + B$ is ample over Z.

We should focus on the case when $-(K_X + B)$ is “non-positive".

Boundedness of complements

For $\mathfrak{R} \subset [0, 1]$, define

$$\Phi(\mathfrak{R}) = \left\{1 - \frac{r}{m} \mid r \in \mathfrak{R}, m \in \mathbb{N}\right\}$$

Theorem ([B-1])

Let $d \in \mathbb{N}$ and $\mathfrak{R} \subset [0, 1]$ be a finite set of rational numbers. Then there is $n \in \mathbb{N}$ satisfying: assume (X, B) is a pair and $X \to Z$ a contraction such that

- *(X, B) is lc of dimension d,*
- *the coefficients of B are in $\Phi(\mathfrak{R})$,*
- *X is Fano type over Z, and*
- *$-(K_X + B)$ is nef over Z.*

Then for any $z \in Z$, there is a strong n-complement $K_X + B^+$ of $K_X + B$ over z.

Boundedness of complements: proof

For simplicity we look at the global case, i.e. Z is a point.

Modifying (X, B), we can reduce the problem to one of the following cases:

1. B has a component S with coeff 1 and $-(K_X + B)$ nef and big, or
2. $K_X + B \equiv 0$ along a fibration $f\colon X \to T$, or
3. (X, B) is *exceptional*.

Exceptional means for any $0 \leq P \sim_{\mathbb{R}} -(K_X + B)$, the pair $(X, B + P)$ is klt.

Those cases require very different inductive treatment.

Case (1): First apply *divisorial adjunction* to define

$$K_S + B_S = (K_X + B)|_S.$$

Boundedness of complements: proof

We can show S is Fano type.

The coefficients of B_S are in a set $\Phi(\mathfrak{S})$ for some fixed finite set $\mathfrak{S}$.

By induction $K_S + B_S$ has a strong n-complement for a bounded n.

The idea then is to lift the complement to X using vanishing theorems.

In the simplest case when (X, B) is log smooth and $B = S$, we look at the exact sequence

$$H^0(-n(K_X + B)) \to H^0(-n(K_X + B)|_S) \to H^1(-n(K_X + B) - S) = 0$$

The vanishing follows from Kawamata-Viehweg vanishing theorem noting that

$$-n(K_X + B) - S = K_X - n(K_X + B) - (K_X + B) = K_X - (n+1)(K_X + B).$$

Boundedness of complements: proof

Since $K_S + B_S$ has a strong n-complement, the middle space in the sequence is non-trivial.

Lifting the corresponding section gives a strong n-complement for $K_X + B$.

Case (2): Apply the *canonical bundle formula* to write

$$K_X + B \sim_{\mathbb{R}} f^*(K_T + B_T + M_T)$$

where B_T is the *discriminant divisor* and M_T is the *moduli divisor*.

The coefficients of B_T happen to be in a set $\Phi(\mathfrak{S})$ for some fixed finite set $\mathfrak{S}$, and pM_T is integral for some bounded number $p \in \mathbb{N}$.

We want to find a complement for $K_T + B_T + M_T$ and pull it back to X.

There is a serious issue: $(T, B_T + M_T)$ is not a pair but a **generalised pair**.

Boundedness of complements: proof

Thus we actually need to prove boundedness of complements for generalised pairs.

This makes life a lot more difficult but fortunately everything turns out to work.

We find a bounded complement for $K_T + B_T + M_T$ and pull it back to get a complement for $K_X + B$.

Case (3): In this case we use effective birationality.

The exceptionality condition implies (X, B) is ϵ-lc for some fixed $\epsilon > 0$.

For simplicity assume $B = 0$ and that X is a Fano variety.

Also assume we already know the effective birationality theorem.

Boundedness of complements: proof

Then there is a bounded $m \in \mathbb{N}$ such that $|-mK_X|$ defines a birational map.

Pick $M \in |-mK_X|$ and let $B^+ = \frac{1}{m}M$.

Since X is exceptional, (X, B^+) is automatically klt, hence $K_X + B^+$ is a strong m-complement.

In practice, boundedness of complements and effective birationality are proved together.

Singularities: boundedness of lc thresholds

We will use an easy version of:

Theorem ([B-2])

For each $d, r \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$ there is $t \in \mathbb{R}^{>0}$ such that if

- *(X, B) is projective ϵ-lc of dimension d,*
- *A is very ample with $A^d \leq r$, and*
- *$A - B$ is ample,*

then

$$\mathrm{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

Effective birationality: proof

Lets recall the theorem.

Theorem ([B-1])

For each $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$ there is $m \in \mathbb{N}$ such that if

- *X is Fano, and ϵ-lc of dimension d,*

then $|-mK_X|$ defines a birational map.

Let $m \in \mathbb{N}$ be the smallest number such that $|-mK_X|$ defines a birational map.

We want to show that m is bounded from above.

Let $n \in \mathbb{N}$ be a number such that $\mathrm{vol}(-nK_X) > (2d)^d$.

Initially we take n to be the smallest such number but we will modify it during the proof.

Effective birationality: proof

The idea is first to show that $\frac{m}{n}$ is bounded from above, and then at the end show that m is bounded.

Applying a standard technique we create a family $\mathcal{G}$ of subvarieties of X such that

if $x, y \in X$ are general points,

then there is $0 \leq \Delta \sim_{\mathbb{Q}} -(n+1)K_X$ and $G \in \mathcal{G}$ such that

(X, Δ) is lc at x with the unique non-klt centre G, and (X, Δ) is not klt at y.

Assume $\dim G = 0$ for all G.

Then $G = \{x\}$ is an isolated non-klt centre.

We can use multiplier ideals and vanishing theorems to lift sections from G.

Effective birationality: proof

We can show $|-nK_X|$ defines a birational map after replacing n with a bounded multiple,

So $\frac{m}{n}$ is bounded from above in this case.

Now lets assume all G have positive dimension.

If $\mathrm{vol}(-mK_X|_G)$ is large, we can decrease dim G.

We then assume $\mathrm{vol}(-mK_X|_G)$ is bounded from above.

By similar arguments, enough to show $\mathrm{vol}(-nK_X|_G)$ is bounded from below.

Showing this lower boundedness is the hard part.

Here we use a kind of adjunction formula.

Effective birationality: proof

If F is the normalisation of G, then we can assume

$$(K_X + \Delta)|_F \sim_{\mathbb{R}} K_F + \Theta_F + P_F$$

where Θ_F is a boundary divisor with coefficients in a fixed DCC set Ψ,

and P_F is big and effective. [Hacon-McKernan-Xu]

Difficulty with applying induction: F may not be Fano, in fact, it can be any type of variety.

Another issue: singularities of $(F, \Theta_F + P_F)$ can be pretty bad.

Idea: use birational boundedness.

Recall: $\mathrm{vol}(-mK_X|_G)$ is bounded from above.

Effective birationality: proof

From this we can deduce G is birational to a bounded model.

That is, there is a bounded projective log smooth pair $(\overline{F}, \Sigma_{\overline{F}})$ and a birational map $\overline{F} \dashrightarrow F$ such that

$\Sigma_{\overline{F}}$ is reduced containing the exceptional divisor of $\overline{F} \dashrightarrow F$ and the support of $\Theta_{\overline{F}}^{\sim}$ (and other relevant divisors).

Surprisingly, the worse the singularities of $(F, \Theta_F + P_F)$ the better.

Because we then produce divisors on $\overline{F}$ with bounded “degree" but with arbitrarily small lc thresholds.

And this contradicts a baby version of the theorem on boundedness of lc thresholds.

Indeed assume $(F, \Theta_F + P_F)$ is not klt.

Effective birationality: proof

A careful study of the adjunction formula allows to write

$$K_F + \Lambda_F := K_X|_F$$

where $\Lambda_F \leq \Theta_F$ and (F, Λ_F) is sub-ϵ-lc.

Put $I_F = \Theta_F + P_F - \Lambda_F$.

Then

$$I_F = K_F + \Theta_F + P_F - K_F - \Lambda_F \sim_{\mathbb{R}} (K_X + \Delta)|_F - K_X|_F = \Delta|_F \sim_{\mathbb{R}} -(n+1)K_X|_F.$$

Moreover, $K_F + \Lambda_F + I_F$ is ample.

Let $\phi \colon F' \to F$ and $\psi \colon F' \to \overline{F}$ be a common resolution.

Let

$$K_{\overline{F}} + \Lambda_{\overline{F}} + I_{\overline{F}} := \psi_* \phi^* (K_F + \Lambda_F + I_F).$$

Effective birationality: proof

Then the above ampleness gives

$$\phi^*(K_F + \Lambda_F + I_F) \leq \psi^*(K_{\overline{F}} + \Lambda_{\overline{F}} + I_{\overline{F}})$$

This implies $(\overline{F}, \Lambda_{\overline{F}} + I_{\overline{F}})$ is not sub-klt.

From this one deduces that $(\overline{F}, \Gamma_{\overline{F}} + I_{\overline{F}})$ is not klt where $\Gamma_{\overline{F}} = (1-\epsilon)\Sigma_{\overline{F}}$.

We can argue that the "degree" of $I_{\overline{F}}$ is arbitrarily small if $\mathrm{vol}(-nK_X|_G)$ is arbitrarily small,

and this contradicts the theorem on boundedness of lc thresholds.

If singularities of $(F, \Theta_F + P_F)$ are good, then we again face some serious difficulties.

Effective birationality: proof

Very roughly, we lift sections from F to X and use this section to modify Δ so that

$(F, \Theta_F + P_F)$ has bad singularities, and we argue as before.

This then shows that $\frac{m}{n}$ is bounded from above.

Finally, we still need to bound m.

This can be done by arguing that $\mathrm{vol}(-mK_X)$ is bounded from above and use this to show X is birationally bounded, and then work on the bounded model.