Linear systems and Fano varieties: introduction

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References:
[B-2] Singularities of linear systems and boundedness of Fano varieties.
We work over an algebraically closed field $k$ of characteristic zero.

Let $X$ be a projective variety with "good" singularities (klt).

We say $X$ is\[\begin{cases} 
\text{Fano} & \text{if } K_X \text{ is anti-ample, eg } \mathbb{P}^n \\
\text{Calabi-Yau} & \text{if } K_X \text{ is trivial, eg abelian varieties} \\
\text{canonically polarised} & \text{if } K_X \text{ is ample}
\end{cases}\]

Such varieties are very interesting in

- birational/algebraic geometry (eg see below; derived categories),
- moduli theory (eg, see below; varieties of general type; Hodge theory),
- differential geometry (eg, Kähler-Einstein metrics, stability),
- arithmetic geometry (eg, existence and density of rational points),
- mathematical physics (eg, string theory, mirror symmetry).
Conjecture (Minimal model and abundance)

Each variety $W$ is birational to a projective variety $Y$ with good singularities such that either

- $Y$ admits a Fano fibration, or
- $Y$ admits a Calabi-Yau fibration, or
- $Y$ is canonically polarised.

Known cases:

- dimension 2: (Castelnuovo, Enriques)(Zariski, Kodaira, etc) 1900,
- dimension 3 (Kawamata, Kollár, Miyaoka, Mori, Reid, Shokurov)(Fano, Hironaka, Iitaka, Iskovskikh-Manin, etc) 1970’s-1990’s,
- any dimension for $W$ of general type (BCHM=B-Cascini-Hacon-McKernan, after Shokurov, etc) 2006.
Minimal model program

How to find such $Y$?

Run the MMP giving a sequence of birational transformations

$$W = W_1 \xrightarrow{\text{div contraction}} W_2 \xrightarrow{\text{flip}} W_3 \rightarrow \cdots \rightarrow W_t = Y$$

The required contractions [Kawamata, Shokurov] and flips [BCHM] exist.

Important ingredient: the $k$-algebra

$$R = \bigoplus_{m \geq 0} H^0(mK_W)$$

is finitely generated [BCHM].

Conjecture

- **Termination**: the program stops after finitely many steps
- **Abundance**: if $K_Y$ not ample, then $Y$ is fibred by Fano’s or CY’s.
Pluricanonical systems and Kodaira dimension

Let $W$ be a smooth projective variety.

The spaces $H^0(W, mK_W)$, for $m \in \mathbb{Z}$, and the linear systems
\[ |mK_W| = \{D \geq 0 \mid D \sim mK_W \} \]
are of great importance.

When $\dim W = 1$, $|K_W|$ determines its geometry to a large extent. The genus $g(W) = h^0(W, K_W)$ is encoded in $|K_W|$.

Moreover, if $g(W) \geq 2$, then $|K_W|$ is base point free, and if also $W$ is not hyperelliptic, then $|K_W|$ defines an embedding of $X$ into a projective space of dimension $g - 1$.

In higher dimension, however, $|K_W|$ often says little about $W$. We need to study $|mK_W|$ for all $m \in \mathbb{Z}$. 
We are led to the notion of **Kodaira dimension** $\kappa(K_W)$, an important birational invariant of $W$.

$$\kappa(K_W) = \begin{cases} \max \{ \dim \phi_{|mK_W|}(W) \} & \text{if } |mK_W| \neq \emptyset \text{ for some } m \in \mathbb{N} \\ -\infty & \text{if } |mK_W| = \emptyset \text{ for all } m \in \mathbb{N} \end{cases}$$

Assume $\kappa(W) \geq 0$.
When $m > 0$ is sufficiently divisible, $|mK_W|$ defines the so-called **Iitaka fibration** $W \dashrightarrow X$ (defined up to birational equivalence).

Here $\kappa(K_W) = \dim X$.

**Conjecture (Effectivity of Iitaka fibration)**

Assume $\kappa(W) \geq 0$. Then there exists $m \in \mathbb{N}$ depending only on $\dim W$ such that $|mK_W|$ defines the Iitaka fibration.
Pluricanonical systems and Kodaira dimension

If $W$ is of general type, i.e. $\kappa(W) = \dim W$, then the conjecture is already known [Hacon-McKernan][Takayama].

Note: $W$ is birational to its canonical model $X$ [BCHM] which is a canonically polarised variety and understanding $|mK_W|$ is the same as understanding $|mK_X|$. 

Now assume $0 \leq \kappa(W) < \dim W$. Then the conjecture is true if we bound certain invariants of the general fibres of the Iitaka fibration [B-Zhang].

Roughly, [B-Zhang] says the conjecture holds if we understand the case $\kappa(W) = 0$.

Note: when $\kappa(W) = 0$, $W$ is (conjecturally) birational to a Calabi-Yau variety, and understanding $|mK_W|$ is the same as understanding such systems on the Calabi-Yau variety.
Now assume $\kappa(W) = -\infty$. Then $|mK_W| = \emptyset$, for $m > 0$.

Conjecturally, $W$ is birational to a variety admitting a Fano fibration.

So it is natural to focus on Fano’s $F$ and study $|-mK_F|$, for $m > 0$, in detail.

There is $m > 0$ depending only on dim $F$ such that $|-mK_F| \neq \emptyset$ [B-1].

Moreover, if we bound the singularities of $F$, then $|-mK_F|$ defines a birational map [B-1].

In fact, in this case the set of such $F$ forms a bounded family.
Let $X$ be a variety with klt singularities, and $f : X \to Z$ be a contraction (surjective projective morphism with connected fibres).

We say $X$ is **Fano over $Z$** if $-K_X$ is ample over $Z$.

**Global case:** $\dim Z = 0$. Here $X$ is a usual Fano variety.

**Fibration case:** $0 < \dim Z < \dim X$. Here $f$ is a Fano fibration whose general fibres are usual Fano’s, e.g. Mori fibre spaces.

**Birational case:** $\dim Z = \dim X$.
- subcase: $f$ flipping or divisorial contraction.
- subcase: $f$ identity. Here we are looking at the germ of a point on a variety (**singularity theory**).
Let $X$ be a normal variety and $X \to Z$ a contraction. (May not be Fano)

Let $z \in Z$.

An $n$-complement of $K_X$ over $z$ is of the form $K_X + \Delta$ such that over $z$ we have

$$\begin{cases} 
(X, \Delta) \text{ has lc singularities}, \\
n(K_X + \Delta) \sim 0
\end{cases}$$

Note that $n\Delta \sim -nK_X$ over $z$. So $n\Delta \in | -nK_X |_z$.

So an $n$-complement over $z$ is an element of $| -nK_X |_z$ with good singularities.
Theorem ([B-1])

For each $d \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that if

- $X$ is Fano over $Z$, of dimension $d$, and
- $z \in Z$,

then there is an $n$-complement of $K_X$ over $z$.

This was conjectured by Shokurov (mid 1990’s, originates in 1970’s).

Proved in dimension 2 by Shokurov.

Partially proved in dimension 3 by Prokhorov-Shokurov.

Example: $X = \mathbb{P}^1$, $\Delta = x_1 + x_2$ with $x_i$ distinct points, then $K_X + \Delta$ is a 1-complement.

Example: $X$ toric Fano, then can take $n = 1$ and $\Delta =$ sum of torus-invariant divisors.
Boundedness of complements

Example: $X$ a smooth Fano 3-fold. Then $|-K_X|$ contains a smooth element, so $K_X$ has a 1-complement. [Shokurov, 70’s]

Example: $X \to Z$ flipping contracting one smooth curve, $X$ a 3-fold with terminal singularities, $z \in Z$. Then $K_X$ has a 1-complement, \textit{analytically}, over $z$. [Mori]

This is used in Mori’s theorem on existence of 3-fold flips.

Remark: assume $X \to Z$ is identity and $x = z \in X$.
In general, Cartier index of $K_X$ near $x$ is not bounded.
But Cartier index of the complement $K_X + \Delta$ near $x$ is bounded.
Boundedness of complements

Example: $X$ a surface, $X \to Z$ identity. ($x = z \in X$)

If $x \in X$ is smooth, then $K_X$ is a 1-complement of itself ($\Delta = 0$).

In the singular case, from classification of singularities we get:

if $x \in X$ is a type

\[
\begin{cases} 
A \text{ singularity, then } K_X \text{ has a 1-complement.} \\
D \text{ singularity, then } K_X \text{ has a 2-complement.} \\
E_6 \text{ singularity, then } K_X \text{ has a 3-complement.} \\
E_7 \text{ singularity, then } K_X \text{ has a 4-complement.} \\
E_8 \text{ singularity, then } K_X \text{ has a 6-complement.}
\end{cases}
\]

[Shokurov]
Effective birationality

**Theorem ([B-1])**

For each $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^>0$ there is $m \in \mathbb{N}$ such that if

- $X$ is Fano, and $\epsilon$-lc of dimension $d$,

then $|-mK_X|$ defines a birational map.

The theorem also holds for relative Fano’s.

Note that $\text{vol}(-K_X) \geq \frac{1}{md}$.

So $m$ depends on both $d, \epsilon$ because $\text{vol}(-K_X)$ can be arbitrarily small for Fano’s even when $d = 2$.

Proved independently by Cascini-McKernan when $\epsilon = 1$, and by Jiang for $d = 3$, by different methods.
Theorem ([B-2])

Assume $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$. Then the set

$$\{X \mid X \text{ $\epsilon$-lc Fano of dimension } d\}$$

forms a bounded family.

Known as Borisov-Alexeev-Borisov conjecture (from early 1990’s).

Previously known partial cases:

- $d = 2$: Alexeev (reproved by Alexeev-Mori),
- toric case: Borisov brothers,
- smooth case: Kollár-Miyaoka-Mori, Nadel, (Fano),
- $d = 3$ and $\epsilon \geq 1$: Kawamata, Kollár-Miyaoka-Mori-Takagi,
- spherical case: Alexeev-Brion.
- special case: Hacon-McKernan-Xu.
For $n \geq 2$ consider

\[
E \subset W_n \xrightarrow{f} X_n
\]

where $X_n$ is the cone over rational curve of deg $n$, and $W_n$ is blowup of vertex, $E$ is the exceptional curve.

$W_n =$ projective bundle of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$.

$E$ is the section given by the summand $\mathcal{O}_{\mathbb{P}^1}(-n)$.

$X_n$ is obtained from $W_n$ by contracting $E$.

$K_{W_n} + \frac{n-2}{n} E = f^* K_{X_n}$.

$X_n$ is $\frac{2}{n}$-lc Fano (as $n \to \infty$, singularities get deeper).

$\{X_n \mid n \in \mathbb{N}\}$ is not a bounded family.
Example of unbounded family of Fano’s

Consider \((\mathbb{P}^2, \Delta = S + T + R)\) where \(S, T, R\) are the coordinate hyperplanes.

Let \(V_1 \rightarrow \mathbb{P}^2\) be blowup of \(x_0 := S \cap T\).

Let \(V_2 \rightarrow V_1\) be blowup of \(x_1 := S^\sim \cap E_1\) where \(E_1\) is the exceptional divisor of \(V_1 \rightarrow \mathbb{P}^2\) and \(S^\sim\) is birational transform.

Similarly define \(V_n \rightarrow V_{n-1} \rightarrow \cdots\) where in each step we blowup intersection point of \(S^\sim\) and newest exceptional curve.

Run MMP on \(-K_{V_n}\) and let \(X_n\) be the resulting model.

\(X_n\) is a klt toric (weak) Fano.

\(\{X_n \mid n \in \mathbb{N}\}\) is not a bounded family because \(\{V_n \mid n \in \mathbb{N}\}\) is not a bounded family.
Groups: birational automorphisms

For a variety $X$, denote the birational automorphism group by $\text{Bir}(X)$.

A corollary of BAB and work of Prokhorov-Shramov concerns $\text{Bir}(X)$.

**Theorem ([B-2])**

For each $d \in \mathbb{N}$, there is $h \in \mathbb{N}$ such that if $X$ is a rationally connected variety of dimension $d$, then for any finite subgroup $G$ of $\text{Bir}(X)$, there is a normal abelian subgroup $H$ of $G$ of index at most $h$.

This says $\text{Bir}(X)$ is **Jordan**.

In particular, $\text{Cr}_d = \text{Bir}(\mathbb{P}^d)$ is Jordan, answering a question of Serre.

Remark: if $X$ is product of $\mathbb{P}^1$ and an elliptic curve, then $\text{Bir}(X)$ is not Jordan [Zarhin].
Singularities: lc thresholds

Let \((X, B)\) be an lc pair and \(L \geq 0\) be an \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor.

Define the **lc threshold**

\[
\text{lct}(X, B, L) = \sup \{ s \mid (X, B + sL) \text{ is lc} \}
\]

which measures singularities of \(L\) with respect to \((X, B)\).

Example: \(X\) a curve. Then

\[
\text{lct}(X, B, L) = \min \left\{ \frac{1 - \mu_x B}{\mu_x L} \right\}_{x \in X}.
\]

Example: \((X, B + L)\) log smooth. Then

\[
\text{lct}(X, B, L) = \min \left\{ \frac{1 - \mu_D B}{\mu_D L} \right\}_{D \subset X} \text{ prime divisor}.
\]

Example: \(X = \mathbb{P}^2\), \(B = 0\), \(L\) a nodal curve. Then

\[
\text{lct}(X, B, L) = 1.
\]
Let \((X, B)\) be a pair and \(A\) be an \(\mathbb{R}\)-divisor.

Let \(|A|_\mathbb{R} = \{N \mid 0 \leq N \sim_\mathbb{R} A\}\).

Define the **lc threshold**

\[
\text{lct}(X, B, |A|_\mathbb{R}) = \sup\{s \mid (X, B + sN) \text{ is lc for every } 0 \leq N \sim_\mathbb{R} A\}
\]

Example: \(X = \mathbb{P}^d\), \(B = 0\), and \(A = -K_X\), then

\[
\text{lct}(X, B, |A|_\mathbb{R}) = \frac{1}{d + 1}.
\]

Example: \(X \subset \mathbb{P}^d\) is a smooth hypersurface of degree \(r \leq d\), \(B = 0\), and \(A = -K_X\), then

\[
\text{lct}(X, B, |A|_\mathbb{R}) = \frac{1}{d + 1 - r}
\]

[Cheltsov-Shramov].
The threshold is especially important when \( X \) is projective and \( A \) is ample. Because of connections with *stability*.

In that context, the lc threshold is also called *global lc threshold* or \( \alpha \)-*invariant*.

But we are interested in it for different reasons. It plays a crucial role in the proof of BAB, etc.

**Theorem ([B-2])**

*For each \( d, r \in \mathbb{N} \) and \( \epsilon \in \mathbb{R}^>0 \) there is \( t \in \mathbb{R}^>0 \) such that if*

- \( (X, B) \) is projective \( \epsilon \)-lc of dimension \( d \),
- \( A \) is very ample with \( A^d \leq r \), and
- \( A - B \) is ample,

*then*

\[
\text{lct}(X, B, |A|_{\mathbb{R}}) \geq t.
\]