Linear systems and Fano varieties: introduction

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References:
[B-1] Anti-pluricanonical systems on Fano varieties.
[B-2] Singularities of linear systems and boundedness of Fano varieties.

Special varieties

We work over an algebraically closed field k of characteristic zero.

Let X be a projective variety with "good" singularities (klt).

$$\text{We say } X \text{ is } \begin{cases} \textbf{Fano} & \text{if } K_X \text{ is anti-ample, eg } \mathbb{P}^n \\ \textbf{Calabi-Yau} & \text{if } K_X \text{ is trivial, eg abelian varieties} \\ \textbf{canonically polarised} & \text{if } K_X \text{ is ample} \end{cases}$$

Such varieties are very interesting in

- birational/algebraic geometry (eg see below; derived categories),
- moduli theory (eg, see below; varieties of general type; Hodge theory),
- differential geometry (eg, Kähler-Einstein metrics, stability),
- arithmetic geometry (eg, existence and density of rational points),
- mathematical physics (eg, string theory, mirror symmetry).

Minimal models and abundance

Conjecture (Minimal model and abundance)

Each variety W is birational to a projective variety Y with good singularities such that either

Y admits a Fano fibration, or

Y admits a Calabi-Yau fibration, or

Y is canonically polarised.

Known cases:

- dimension 2: (Castelnuovo, Enriques)(Zariski, Kodaira, etc) 1900,
- dimension 3 (Kawamata, Kollár, Miyaoka, Mori, Reid, Shokurov)(Fano, Hironaka, Iitaka, Iskovskikh-Manin, etc) 1970's-1990's,
- any dimension for W of general type (BCHM=B-Cascini-Hacon-M^{c}Kernan, after Shokurov, etc) 2006.

Minimal model program

How to find such Y?

Run the MMP giving a sequence of birational transformations

$$W = W_1 \overset{\text{div contraction}}{\dashrightarrow} W_2 \overset{\text{flip}}{\dashrightarrow} W_3 \dashrightarrow \cdots \dashrightarrow W_t = Y$$

The required contractions [Kawamata, Shokurov] and flips [BCHM] exist.

Important ingredient: the k-algebra

$$R = \bigoplus_{m \geq 0} H^0(mK_W)$$

is finitely generated [BCHM].

Conjecture

- **Termination:** *the program stops after finitely many steps*
- **Abundance:** *if K_Y not ample, then Y is fibred by Fano's or CY's.*

Pluricanonical systems and Kodaira dimension

Let W be a smooth projective variety.

The spaces $H^0(W, mK_W)$, for $m \in \mathbb{Z}$, and the linear systems

$$|mK_W| = \{D \geq 0 \mid D \sim mK_W\}$$

are of great importance.

When $\dim W = 1$, $|K_W|$ determines its geometry to a large extent.

The genus $g(W) = h^0(W, K_W)$ is encoded in $|K_W|$.

Moreover, if $g(W) \geq 2$, then $|K_W|$ is base point free, and if also W is not hyperelliptic, then $|K_W|$ defines an embedding of X into a projective space of dimension $g - 1$.

In higher dimension, however, $|K_W|$ often says little about W. We need to study $|mK_W|$ for all $m \in \mathbb{Z}$.

Pluricanonical systems and Kodaira dimension

We are led to the notion of **Kodaira dimension** $\kappa(K_W)$, an important birational invariant of W.

$$\kappa(K_W) = \begin{cases} \max\{\dim \phi_{|mK_W|}(W)\} & \text{if } |mK_W| \neq \emptyset \text{ for some } m \in \mathbb{N} \\ -\infty & \text{if } |mK_W| = \emptyset \text{ for all } m \in \mathbb{N} \end{cases}$$

Assume $\kappa(W) \geq 0$.
When $m > 0$ is sufficiently divisible, $|mK_W|$ defines the so-called **Iitaka fibration** $W \dashrightarrow X$ (defined up to birational equivalence).

Here $\kappa(K_W) = \dim X$.

Conjecture (Effectivity of Iitaka fibration)

Assume $\kappa(W) \geq 0$. Then there exists $m \in \mathbb{N}$ depending only on $\dim W$ such that $|mK_W|$ defines the Iitaka fibration.

Pluricanonical systems and Kodaira dimension

If W is of *general type*, i.e. $\kappa(W) = \dim W$, then the conjecture is already known [Hacon-McKernan][Takayama].

Note: W is birational to its canonical model X [BCHM] which is a canonically polarised variety and understanding $|mK_W|$ is the same as understanding $|mK_X|$.

Now assume $0 \leq \kappa(W) < \dim W$.
Then the conjecture is true if we bound certain invariants of the general fibres of the Iitaka fibration [B-Zhang].

Roughly, [B-Zhang] says the conjecture holds if we understand the case $\kappa(W) = 0$.

Note: when $\kappa(W) = 0$, W is (conjecturally) birational to a Calabi-Yau variety, and understanding $|mK_W|$ is the same as understanding such systems on the Calabi-Yau variety.

Pluricanonical systems and Kodaira dimension

Now assume $\kappa(W) = -\infty$. Then $|mK_W| = \emptyset$, for $m > 0$.

Conjecturally, W is birational to a variety admitting a Fano fibration.

So it is natural to focus on Fano's F and study $|-mK_F|$, for $m > 0$, in detail.

There is $m > 0$ depending only on dim F such that $|-mK_F| \neq \emptyset$ [B-1].

Moreover, if we bound the singularities of F, then $|-mK_F|$ defines a birational map [B-1].

In fact, in this case the set of such F forms a bounded family.

Relative Fano varieties

Let X be a variety with klt singularities, and $f\colon X \to Z$ be a contraction (surjective projective morphism with connected fibres).

We say X is **Fano over Z** if $-K_X$ is ample over Z.

Global case: $\dim Z = 0$. Here X is a usual Fano variety.

Fibration case: $0 < \dim Z < \dim X$. Here f is a Fano fibration whose general fibres are usual Fano's, e.g. Mori fibre spaces.

Birational case: $\dim Z = \dim X$.

- subcase: f flipping or divisorial contraction.
- subcase: f identity. Here we are looking at the germ of a point on a variety (*singularity theory*).

Complements

Let X be a normal variety and $X \to Z$ a contraction. (May not be Fano)

Let $z \in Z$.

An n-**complement** of K_X over z is of the form $K_X + \Delta$ such that over z we have

$$\begin{cases} (X, \Delta) & \text{has lc singularities,} \\ n(K_X + \Delta) \sim 0 \end{cases}$$

Note that $n\Delta \sim -nK_X$ over z. So $n\Delta \in |-nK_X|_{/z}$.

So an n-complement over z is an element of $|-nK_X|_{/z}$ with good singularities.

Boundedness of complements

Theorem ([B-1])

For each $d \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that if

- *X is Fano over Z, of dimension d, and*
- *$z \in Z$,*

then there is an n-complement of K_X over z.

This was conjectured by Shokurov (mid 1990's, originates in 1970's).

Proved in dimension 2 by Shokurov.

Partially proved in dimension 3 by Prokhorov-Shokurov.

Example: $X = \mathbb{P}^1$, $\Delta = x_1 + x_2$ with x_i distinct points, then $K_X + \Delta$ is a 1-complement.

Example: X toric Fano, then can take $n = 1$ and $\Delta =$ sum of torus-invariant divisors.

Boundedness of complements

Example: X a smooth Fano 3-fold. Then $|-K_X|$ contains a smooth element, so K_X has a 1-complement. [Shokurov, 70's]

Example: $X \to Z$ flipping contracting one smooth curve, X a 3-fold with terminal singularities, $z \in Z$. Then K_X has a 1-complement, *analytically*, over z. [Mori]

This is used in Mori's theorem on existence of 3-fold flips.

Remark: assume $X \to Z$ is identity and $x = z \in X$.
In general, Cartier index of K_X near x is not bounded.

But Cartier index of the complement $K_X + \Delta$ near x is bounded.

Boundedness of complements

Example: X a surface, $X \to Z$ identity. ($x = z \in X$)

If $x \in X$ is smooth, then K_X is a 1-complement of itself ($\Delta = 0$).

In the singular case, from classification of singularities we get:

$$\text{if } x \in X \text{ is a type } \begin{cases} A \text{ singularity, then } K_X \text{ has a } 1\text{-complement.} \\ D \text{ singularity, then } K_X \text{ has a } 2\text{-complement.} \\ E_6 \text{ singularity, then } K_X \text{ has a } 3\text{-complement.} \\ E_7 \text{ singularity, then } K_X \text{ has a } 4\text{-complement.} \\ E_8 \text{ singularity, then } K_X \text{ has a } 6\text{-complement.} \end{cases}$$

[Shokurov]

Effective birationality

Theorem ([B-1])

For each $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$ there is $m \in \mathbb{N}$ such that if

- *X is Fano, and ϵ-lc of dimension d,*

then $|-mK_X|$ defines a birational map.

The theorem also holds for relative Fano's.

Note that $\text{vol}(-K_X) \geq \frac{1}{m^d}$.

So m depends on both d, ϵ because $\text{vol}(-K_X)$ can be arbitrarily small for Fano's even when $d = 2$.

Proved independently by Cascini-McKernan when $\epsilon = 1$, and by Jiang for $d = 3$, by different methods.

Boundedness of Fano varieties: BAB

Theorem ([B-2])

Assume $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$. Then the set

$$\{X \mid X \;\; \epsilon\text{-lc Fano of dimension } d\}$$

forms a bounded family.

Known as Borisov-Alexeev-Borisov conjecture (from early 1990's).

Previously known partial cases:

- $d = 2$: Alexeev (reproved by Alexeev-Mori),
- toric case: Borisov brothers,
- smooth case: Kollár-Miyaoka-Mori, Nadel, (Fano),
- $d = 3$ and $\epsilon \geq 1$: Kawamata, Kollár-Miyaoka-Mori-Takagi,
- spherical case: Alexeev-Brion.
- special case: Hacon-M^{c}Kernan-Xu.

Example of unbounded family of Fano's

For $n \geq 2$ consider

$$\begin{array}{ccc} E \subset W_n & \xrightarrow{f} & X_n \\ \downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

where X_n is the cone over rational curve of deg n, and W_n is blowup of vertex, E is the exceptional curve.

$W_n =$ projective bundle of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$.

E is the section given by the summand $\mathcal{O}_{\mathbb{P}^1}(-n)$.

X_n is obtained from W_n by contracting E.

$K_{W_n} + \frac{n-2}{n} E = f^* K_{X_n}$.

X_n is $\frac{2}{n}$-lc Fano (as $n \to \infty$, singularities get deeper).

$\{X_n \mid n \in \mathbb{N}\}$ is not a bounded family.

Example of unbounded family of Fano's

Consider $(\mathbb{P}^2, \Delta = S + T + R)$ where S, T, R are the coordinate hyperplanes.

Let $V_1 \to \mathbb{P}^2$ be blowup of $x_0 := S \cap T$.

Let $V_2 \to V_1$ be blowup of $x_1 := S^\sim \cap E_1$ where E_1 is the exceptional divisor of $V_1 \to \mathbb{P}^2$ and $S^\sim$ is birational transform.

Similarly define $V_n \to V_{n-1} \to \cdots$ where in each step we blowup intersection point of $S^\sim$ and newest exceptional curve.

Run MMP on $-K_{V_n}$ and let X_n be the resulting model.

X_n is a klt toric (weak) Fano.

$\{X_n \mid n \in \mathbb{N}\}$ is not a bounded family because $\{V_n \mid n \in \mathbb{N}\}$ is not a bounded family..

Groups: birational automorphisms

For a variety X, denote the birational automorphism group by $\mathrm{Bir}(X)$.

A corollary of BAB and work of Prokhorov-Shramov concerns $\mathrm{Bir}(X)$.

Theorem ([B-2])

For each $d \in \mathbb{N}$, there is $h \in \mathbb{N}$ such that if
X is a rationally connected variety of dimension d,
then for any finite subgroup G of $\mathrm{Bir}(X)$, there is a normal abelian subgroup H of G of index at most h.

This says $\mathrm{Bir}(X)$ is *Jordan*.

In particular, $\mathrm{Cr}_d = \mathrm{Bir}(\mathbb{P}^d)$ is Jordan, answering a question of Serre.

Remark: if X is product of $\mathbb{P}^1$ and an elliptic curve, then $\mathrm{Bir}(X)$ is not Jordan [Zarhin].

Singularities: lc thresholds

Let (X, B) be an lc pair and $L \geq 0$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor.

Define the **lc threshold**

$$\operatorname{lct}(X, B, L) = \sup\{s \mid (X, B + sL) \text{ is lc}\}$$

which measures singularities of L with respect to (X, B).

Example: X a curve. Then

$$\operatorname{lct}(X, B, L) = \min\{\frac{1 - \mu_x B}{\mu_x L}\}_{x \in X}.$$

Example: $(X, B + L)$ log smooth. Then

$$\operatorname{lct}(X, B, L) = \min\{\frac{1 - \mu_D B}{\mu_D L}\}_{D \subset X \text{ prime divisor}}.$$

Example: $X = \mathbb{P}^2$, $B = 0$, L a nodal curve. Then

$$\operatorname{lct}(X, B, L) = 1.$$

Singularities: lc thresholds of linear systems

Let (X, B) be a pair and A be an $\mathbb{R}$-divisor.

Let $|A|_{\mathbb{R}} = \{N \mid 0 \leq N \sim_{\mathbb{R}} A\}$.

Define the **lc threshold**

$$\mathrm{lct}(X, B, |A|_{\mathbb{R}}) = \sup\{s \mid (X, B + sN) \text{ is lc for every } 0 \leq N \sim_{\mathbb{R}} A\}$$

Example: $X = \mathbb{P}^d$, $B = 0$, and $A = -K_X$, then

$$\mathrm{lct}(X, B, |A|_{\mathbb{R}}) = \frac{1}{d+1}.$$

Example: $X \subset \mathbb{P}^d$ is a smooth hypersurface of degree $r \leq d$, $B = 0$, and $A = -K_X$, then

$$\mathrm{lct}(X, B, |A|_{\mathbb{R}}) = \frac{1}{d+1-r}$$

[Cheltsov-Shramov].

Singularities: lc thresholds of linear systems

The threshold is especially important when X is projective and A is ample. Because of connections with *stability*.

In that context, the lc threshold is also called *global lc threshold* or α*-invariant*.

But we are interested in it for different reasons. It plays a crucial role in the proof of BAB, etc.

Theorem ([B-2])

For each $d, r \in \mathbb{N}$ *and* $\epsilon \in \mathbb{R}^{>0}$ *there is* $t \in \mathbb{R}^{>0}$ *such that if*

- (X, B) *is projective* ϵ*-lc of dimension* d*,*
- A *is very ample with* $A^d \leq r$*, and*
- $A - B$ *is ample,*

then

$$\operatorname{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$