LOG CANONICAL ALGEBRAS AND MODULES

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Abstract. Let \((X/Z, B)\) be a lc pair with \(K_X + B\) pseudo-effective\(\mathbb{Z}\) and \(Z\) affine. We show that \((X/Z, B)\) has a good log minimal model iff its log canonical algebra and modules are finitely generated.

1. Introduction

Let \(X \to Z\) be a projective morphism of varieties over \(\mathbb{C}\) with \(Z = \text{Spec } A\) being affine. For any Cartier divisor \(L\) on \(X\) we have the graded ring

\[ R(L) := \bigoplus_{m \geq 0} H^0(X, mL) \]

which is a graded \(A\)-algebra. On the other hand, for each \(\mathcal{O}_X\)-module \(\mathcal{F}\) on \(X\) and each integer \(p\), we have the graded \(R(L)\)-module \(M_p(L) = \bigoplus_{m \in \mathbb{Z}} M_m\) where \(M_m = 0\) if \(m < p\) but

\[ M_m = H^0(X, \mathcal{F}(mL)) \]

if \(m \geq p\). Here \(\mathcal{F}(mL)\) stands for \(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(mL)\) and the module structure is given via the pairing

\[ H^0(X, mL) \otimes H^0(X, \mathcal{F}(nL)) \to H^0(X, \mathcal{F}((m + n)L)) \]

If \(\mathcal{F} = \mathcal{O}_X(D)\) for some divisor \(D\) we usually write \(M_p^D(L)\) instead of \(M_{\mathcal{O}_X(D)}^p(L)\).

When \(L = I(K_X + B)\) for a log canonical pair \((X, B)\) and integer \(I > 0\), we refer to \(R(L)\) as a log canonical algebra and refer to the module \(M_p^L(L)\) as a log canonical module. The following theorem is the main result of this short note.

Theorem 1.1. Assume that \((X/Z, B)\) is lc where \(Z = \text{Spec } A\), and let \(I\) be a positive integer so that \(L := I(K_X + B)\) is Cartier. If \(K_X + B\) is pseudo-effective\(\mathbb{Z}\), then the following are equivalent:

1. \((X/Z, B)\) has a good log minimal model;
2. \(R(L)\) is a finitely generated \(A\)-algebra, and for any very ample\(\mathbb{Z}\) divisor \(G\) and integer \(p\) the module \(M_G^p(L)\) is finitely generated over \(R(L)\).

The klt case of the theorem is a result of Demailly-Hacon-Păun [3]. Our proof below is somewhat different and more algebraic in nature, and it also works in the lc case. Note that we have assumed \(Z\) to be affine for simplicity of notation; the general case

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can be formulated and proved in a similar way.

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2. Preliminaries

Varieties are assumed to be over \( \mathbb{C} \) unless stated otherwise. We use the notion and notation of pairs and log minimal models as in [1]. Singularities such as lc, klt, and dlt are as in [5]. We use the numerical Kodaira dimension \( \kappa_{\sigma} \) introduced by Nakayama [6].

Rings are assumed to be commutative with identity. A graded ring is of the form \( R = \bigoplus_{m \geq 0} R_m \), that is, graded by non-negative integers, and a graded module is of the form \( M = \bigoplus_{m \in \mathbb{Z}} M_m \), that is, it is graded by the integers. For an element \((\ldots, 0, \alpha, 0, \ldots)\) of degree \( m \) we often abuse notation and just write \( \alpha \) but keep in mind that \( \alpha \) has degree \( m \).

Remark 2.1 (Truncation principle) (1) Let \( R = \bigoplus_{m \geq 0} R_m \) be a graded ring and \( I \) a positive integer. Define the truncated ring \( R[I] = \bigoplus_{m \geq 0} R'_m \) by putting \( R'_m = R_m \) if \( I | m \) and \( R'_m = 0 \) otherwise. Note that the degree structure is different from the usual definition of truncation. However, it is more convenient for us to define it in this way.

(2) With \( R \) and \( I \) as in (1), assume that \( R_0 \) is a Noetherian ring and that \( R \) is an integral domain. It is well-known that:

\( R \) is a finitely generated \( R_0 \)-algebra iff \( R[I] \) is a finitely generated \( R_0 \)-algebra.

(3) Again \( R \) and \( I \) are as in (1). Let \( M = \bigoplus_{m \in \mathbb{Z}} M_m \) be a graded \( R \)-module. Let \( N_i = \bigoplus_{m \in \mathbb{Z}} N_{m,i} \) where \( N_{m,i} = M_m \) if \( m \equiv i \pmod{I} \) but \( N_{m,i} = 0 \) otherwise. Then, each \( N_i \) is a graded module over \( R[I] \) and we have the decomposition

\[ M \simeq N_0 \oplus N_1 \oplus \cdots \oplus N_{I-1} \]

as graded \( R[I] \)-modules. If the modules \( N_0, \ldots, N_{I-1} \) are finitely generated over \( R[I] \), then \( M \) is also a finitely generated \( R[I] \)-module hence a finitely generated \( R \)-module too.

Theorem 2.2. Let \( X \to Z \) be a projective morphism of normal varieties with \( Z = \text{Spec} \, A \), and let \( L \) be a Cartier divisor on \( X \) such that \( R(L) \) is a finitely generated \( A \)-algebra. Fix an integer \( p \). Then we have:

(1) Assume that \( M^p_G(L) \) is a finitely generated \( R(L) \)-module for any very ample/Z divisor \( G \). Then \( M^p_\mathcal{F}(L) \) is a finitely generated \( R(L) \)-module for every torsion-free coherent sheaf \( \mathcal{F} \).

(2) If \( L \) is big/Z, then \( M^p_\mathcal{F}(L) \) is a finitely generated \( R(L) \)-module for every torsion-free coherent sheaf \( \mathcal{F} \).

(3) Let \( \mathcal{F} \) be a coherent sheaf and \( I > 0 \) an integer. For each \( 0 \leq i < I \), assume that \( M^p_{\mathcal{F}(iL)}(IL) \) is a finitely generated \( R(IL) \)-module where \( q_i \in \mathbb{Z} \) is the smallest number satisfying \( q_i I + i \geq p \). Then \( M^p_\mathcal{F}(L) \) is a finitely generated \( R(L) \)-module.
Proof. (1) Let $G$ be a very ample $\mathbb{Z}$ divisor and pick a reflexive coherent sheaf $\mathcal{F}$. There is a surjective morphism $\bigoplus_{j=1}^{r} \mathcal{O}_X(-l_jG) \to \mathcal{F}^\vee$ for some $l_j > 0$ where $\vee$ stands for dual. Taking the dual of this morphism gives an injective morphism

$$\mathcal{F} \simeq \mathcal{F}^{\vee \vee} \to \mathcal{E} = \bigoplus_{j=1}^{r} \mathcal{O}_X(l_jG)$$

which in turn gives an injective map $M^p_\mathcal{F}(L) \to M^p_\mathcal{F}(L)$. By assumptions, $M^p_\mathcal{F}(L)$ is finitely generated over $R(L)$ which in particular means that $M^p_\mathcal{F}(L)$ is Noetherian as $R(L)$ is Noetherian. Therefore, each submodule of $M^p_\mathcal{F}(L)$ is also finitely generated over $R(L)$, in particular, $M^p_\mathcal{F}(L)$.

Now assume that $\mathcal{F}$ is just a torsion-free coherent sheaf. The natural morphism $\mathcal{F} \to \mathcal{F}^{\vee \vee}$ is injective (cf. [4]). So, we get an injective map $M^p_\mathcal{F}(L) \to M^p_\mathcal{F}^\vee(L)$ and the claim follows since $\mathcal{F}^{\vee \vee}$ is a reflexive sheaf.

(2) By (1), it is enough to verify the finite generation of $M^p_\mathcal{F}(L)$ for very ample $\mathbb{Z}$ divisors $G$. Since $L$ is big $\mathbb{Z}$, there is $n > 0$ such that $nL \sim E + G$ for some effective Cartier divisor $E$. Thus, there is an injective map

$$M^p_\mathcal{F}(L) \to M^p_{E+G}(L) \simeq M^p_{nL}(L)$$

So, it is enough to show that $M^p_{nL}(L)$ is a finitely generated $R(L)$-module. This in turn follows from finite generation of $M^p_{nL}(L)$. Now the elements of degree $-n$ are $H^0(X, -nL + nL) = H^0(X, \mathcal{O}_X)$ which contains $1 \in \mathcal{O}_X(X)$. If $\alpha \in M^{-n}_{nL}(L)$ is a homogeneous element of degree $m \geq -n$, that is, an element of $H^0(X, mL + nL)$, then $\alpha = \alpha \cdot 1$ where we consider the second $\alpha$ as an element of $R(L)$ of degree $m + n$ and we consider $1$ as an element of $M^{-n}_{nL}(L)$ of degree $-n$. So, $M^{-n}_{nL}(L)$ is generated over $R(L)$ by the element $1$ of degree $-n$.

(3) We can write $M^p_{\mathcal{F}}(L) \simeq N_0 \oplus N_1 \oplus \cdots \oplus N_{i-1}$ as in Remark 2.1 (3). Let $N'_i$ be the module over $R(IL)$ whose $m$-th degree summand is just $N_{mL+i} = M_{mL+i}$. In fact, $N'_i = M^q_{\mathcal{F}(IL)}(IL)$ where $q_i \in \mathbb{Z}$ is the smallest number satisfying $q_iL + i \geq p$. Note that the degree $n$ elements of $R(IL)$ are the same as the degree $nL$ elements of $R(L)[i]$, and the degree $m$ elements of $N'_i$ are the same as the degree $mL + i$ elements of $N_i$. By assumptions, $N'_i$ is a finitely generated $R(IL)$-module. Therefore, $N_i$ is a finitely generated $R(L)[i]$-module, and so by Remark 2.1 (3) we are done. \hfill \square

3. Proof of Theorem 1.1

Throughout this section we let $X \to Z$ be a projective morphism of normal varieties over $\mathbb{C}$ with $Z = \text{Spec } A$.

Lemma 3.1. Assume that $Z = \text{pt}$ and $L$ is a Cartier divisor on $X$. Further assume that for any very ample divisor $G$ the module $M^0_G(L)$ is finitely generated over $R(L)$. Then, $\kappa(L) = \kappa_\sigma(L)$.

Proof. The inequality $\kappa(L) \leq \kappa_\sigma(L)$ follows from the fact that $\kappa(L) = \kappa(JL)$ and $\kappa_\sigma(L) = \kappa_\sigma(JL)$ for any positive integer $J$ and the fact that for some $J$ and certain
constants \(c_1, c_2 > 0\) we have
\[
c_1 m^{\kappa(L)} \leq h^0(X, mJL) \leq c_2 m^{\kappa(L)}
\]
for any \(m \gg 0\).

For the converse \(\kappa(L) \geq \kappa_\sigma(L)\), we may assume that \(\kappa_\sigma(L) \geq 0\) and we can choose a very ample divisor \(G\) so that \(\kappa_\sigma(L)\) satisfies
\[
\limsup_{m \to +\infty} \frac{h^0(X, mL + G)}{m^{\kappa_\sigma(L)}} > 0
\]
By assumptions, \(M^0_\sigma(L)\) is a finitely generated \(R(L)\)-module. Let \(\{\alpha_1, \ldots, \alpha_r\}\) be a set of generators of homogeneous elements with \(n_i := \deg \alpha_i\). For any \(\alpha \in M^0_\sigma(L)\) of degree \(m\), there are homogeneous elements \(a_i \in R(L)\) such that \(\alpha = \sum_i a_i \alpha_i\). It is clear that \(\deg a_i = m - n_i\). Thus,
\[
h^0(X, (m - n_1)L) + \cdots + h^0(X, (m - n_r)L) \geq h^0(X, mL + G)
\]
which implies that
\[
\limsup_{m \to +\infty} \frac{h^0(X, (m - n_1)L) + \cdots + h^0(X, (m - n_r)L)}{m^{\kappa_\sigma(L)}} > 0
\]
hence \(\kappa(L) \geq \kappa_\sigma(L)\).

The next result is well-known but we include its proof for convenience.

**Lemma 3.2.** Let \(L\) be a Cartier divisor on \(X\) with \(h^0(X, nL) \neq 0\) for some \(n > 0\). Then, the following are equivalent:

1. \(R(L)\) is a finitely generated \(A\)-algebra;
2. there exist a projective birational morphism \(f : W \to X\) from a smooth variety, a positive integer \(J\), and Cartier divisors \(E\) and \(F\) such that \(|F|\) is base point free, and

\[
\text{Mov } f^*mJL = mF \text{ and } \text{Fix } f^*mJL = mE
\]
for every positive integer \(m\).

**Proof.** Assume that \(R(L)\) is a finitely generated \(A\)-algebra. Perhaps after replacing \(L\) with \(JL\) for some positive integer \(J\), we may assume that the algebra \(R(L)\) is generated by elements \(\alpha_1, \ldots, \alpha_r\) of degree 1, and that there is a resolution \(f : W \to X\) on which \(f^*L = F + E\) where \(F\) is free, \(\text{Mov } f^*L = F\), and \(\text{Fix } f^*L = E\). We could in addition assume that \(F \geq 0\) with no common components with \(E\). Obviously, \(\text{Fix } mf^*L \leq mE\) for any \(m > 0\). Suppose that equality does not hold for some \(m > 0\). Take \(m > 0\) minimal with this property. Since \(E = \text{Fix } f^*L, m > 1\). There is \(\alpha \in H^0(W, mf^*L)\) and a component \(S\) of \(E\) such that \(\mu_S(\alpha) < 0\) where \(\mu\) stands for multiplicity, that is, the coefficient and \((\alpha)\) is the divisor associated to the rational function \(\alpha\). By assumptions, \(\alpha = \sum a_i \alpha_i\) where \(a_i\) are elements of \(H^0(W, (m - 1)f^*L)\). Thus,
\[
\mu_S(\alpha) \geq \min \{\mu_S(a_i) + \mu_S(\alpha_i)\} = \mu_S(a_j) + \mu_S(\alpha_j)
\]
for some \(j\). The choice of \(m\) ensures that \(\mu_S(\alpha_j) \geq 0\) and \(\mu_S(\alpha_j) \geq 0\). This contradicts \(\mu_S(\alpha) < 0\).
Conversely, assume that there exist \( f : W \to X, J, E, \) and \( F \) as in the theorem. Then, \( R(JL) \simeq R(f^* JL) \simeq R(F) \) is a finitely generated \( A \)-algebra as \( |F| \) is base point free. This implies that \( R(L) \) is a finitely generated \( A \)-algebra by Remark 2.1.

**Lemma 3.3.** Let \( L \) be a Cartier divisor on \( X \) with \( h^0(X, nL) \neq 0 \) for some \( n > 0 \) and with \( R(L) \) a finitely generated \( A \)-algebra. Assume further that \( M^0_G(L) \) is a finitely generated \( R(L) \)-module for any very ample/Z divisor \( G' \). Let \( f, W, F, E, J \) be as in Lemma 3.2. Fix a nonnegative integer \( r \) and a very ample/Z divisor \( G \) on \( W \). Then,

\[
\operatorname{Supp} \operatorname{Fix}(m(f^* JL + rF) + G) = \operatorname{Supp} E
\]

for every integer \( m \gg 0 \).

**Proof.** Let \( G'' \) be a very ample/Z divisor on \( W \) and let \( G' \) be a very ample/Z divisor on \( X \) such that \( G'' \leq f^* G' \). By assumptions, \( R(L) \) is a Noetherian ring and \( M = M^0_G(L) \) is a Noetherian \( R(L) \)-module. Moreover, \( R(L) \) is integral over the ring \( R(L)[^J] \) which implies that \( M \) is a finitely generated \( R(L)[^J] \)-module. Put \( N_0 = \bigoplus_{m \geq 0} N_{m,0} \) where \( N_{m,0} = \emptyset \) if \( J|m \) but \( N_{m,0} = \emptyset \) otherwise, as in Remark 2.1 (3). Since \( N_0 \) is an \( R(L)[^J] \)-submodule of \( M \), it is finitely generated over \( R(L)[^J] \). This corresponds to saying that \( M^0_G(JL) \) is a finitely generated \( R(JL) \)-module. On the other hand, \( M^0_G(f^* JL) \) is a submodule of \( M^0_G(f^* JL) \) hence a finitely generated \( R(f^* JL) \)-module. Thus, after replacing \( L \) with \( f^* JL \) and \( X \) with \( W \) we can assume that \( J = 1 \) and \( W = X \). We may also assume that \( F, G \geq 0 \) and that \( F + G \) has no common component with \( E \).

Obviously,

\[
\operatorname{Supp} \operatorname{Fix}(m(L + rF) + G) \subseteq \operatorname{Supp} E
\]

for every integer \( m > 0 \). Assume that there is a component \( S \) of \( E \) which does not belong to \( \operatorname{Supp} \operatorname{Fix}(m(L + rF) + G) \) for some \( m > 0 \). Let \( \alpha \in H^0(X, m(L + rF) + G) \) so that

\[
S \not\subseteq \operatorname{Supp}((\alpha) + m(L + rF) + G)
\]

which in particular means that \( \mu_S(\alpha) = -m \mu_S E \). Since

\[
(m + mr)L + G = m(L + rF) + G + mrE
\]

and \( mrE \geq 0 \), there is \( \alpha' \) in \( M^0_G(L) \) of degree \( m + mr \) such that \( \alpha' = \alpha \) as rational functions on \( X \).

Assume that \( \{\alpha_1, \ldots, \alpha_r\} \) is a set of homogeneous generators of \( M^0_G(L) \) with \( n_i := \deg \alpha_i \). We can write \( \alpha' = \sum a_i \alpha_i \) where \( a_i \in R(L) \) is homogenous of degree \( m + mr - n_i \). Therefore,

\[
\mu_S(\alpha') \geq \min \{\mu_S(a_i) + \mu_S(\alpha_i)\}
\]

Since

\[
\operatorname{Fix}(m + mr - n_i)L = (m + mr - n_i)E
\]

we have \( \mu_S(a_i) \geq 0 \) hence if the above minimum is attained at index \( j \), then

\[
-m \mu_S E = \mu_S(\alpha) = \mu_S(\alpha') \geq \mu_S(\alpha_j)
\]

from which we get \( m \mu_S E \leq -\mu_S(\alpha_j) \). This means that such \( m \) cannot be too large so the theorem holds for \( m \gg 0 \). \( \square \)
Proof. (of Theorem 1.1) \(1 \implies 2\): Assume that \((X/Z, B)\) has a good log minimal model \((Y/Z, B_Y)\). By Theorem 2.2, we can replace \(I\) with a multiple so that we can assume that \(|I(K_Y + B_Y)|\) is base point free. Let \(f: W \to X\) and \(g: W \to Y\) be a common resolution. Then, we can write

\[f^*I(K_X + B) = g^*I(K_Y + B_Y) + E\]

where \(E \geq 0\) and exceptional/\(Y\) \([1, \text{Remark 2.4}]\). Then, by letting \(L_Y := I(K_Y + B_Y)\) we have \(R(L) \cong R(L_Y)\) as \(A\)-algebras and this is a finitely generated \(A\)-algebra as \(|L_Y|\) is base point free by assumptions. Let \(\mathcal{G}\) be any torsion-free coherent sheaf on \(Y\) and let \(\pi: Y \to T/Z\) be the contraction defined by \(|L_Y|\). There is an ample/\(Z\) divisor \(N\) on \(T\) such that \(L_Y \sim \pi^*N\). Then, by the projection formula

\[\pi_* (\mathcal{G}(mL_Y)) \cong (\pi_* \mathcal{G})(mN)\]

hence

\[H^0(Y, \mathcal{G}(mL_Y)) \cong H^0(T, (\pi_* \mathcal{G})(mN))\]

So \(R(L) \cong R(L_Y) \cong R(N)\) as \(A\)-algebras and \(M^p_{\mathcal{G}}(L_Y) \cong M^p_{\mathcal{G}}(N)\) as modules. By Theorem 2.2 (2), \(M^p_{\mathcal{G}}(N)\) is a finitely generated \(R(N)\)-module hence \(M^p_{\mathcal{G}}(L_Y)\) is a finitely generated \(R(L_Y)\)-module.

Next we prove the finite generation of \(M^p_{\mathcal{G}}(L)\) for any coherent torsion-free sheaf \(\mathcal{F}\) on \(X\). By Theorem 2.2 (1), we may assume that \(\mathcal{F} = \mathcal{O}_X(G)\) where \(G\) is some very ample/\(Z\) divisor. For each \(m\) we have an isomorphism

\[H^0(X, mL + G) \cong H^0(W, f^*mL + f^*G)\]

and this is isomorphic to a subspace of \(H^0(Y, mL_Y + g_*f^*G)\). So, \(M^p_{\mathcal{G}}(L)\) is isomorphic to a submodule of \(M^p_{\mathcal{G}}(L_Y)\). Therefore, \(M^p_{\mathcal{G}}(L)\) is a finitely generated \(R(L)\)-module because \(M^p_{\mathcal{G}}(L_Y)\) is a finitely generated \(R(L_Y)\)-module and \(R(L_Y)\) is Noetherian.

\(2 \implies 1\): We may assume that \(X \to Z\) is a contraction. Let \(V\) be the generic fibre of \(X \to Z\), and let \(K\) be the function field of \(Z\). As \(Z\) is affine, by base change theorems, \(R(L|_V) \cong R(L) \otimes_A K\) is a finitely generated \(K\)-algebra, and for any very ample/\(Z\) divisor \(G\) on \(X\) the module \(M^0_{G|_V}(L|_V) \cong M^0_G(L) \otimes_A K\) is finitely generated over \(R(L|_V)\). By Theorem 2.2 and Lemma 3.1, \(\kappa(L|_V) \geq 0\) which in particular implies that \(h^0(X, nL) \neq 0\) for some \(n > 0\).

Let \(f, W, E, F, J\) be as in Theorem 3.2. We may assume that \(f\) gives a log resolution of \((X/Z, B)\). Let \(B_W\) be \(B^\sim\) plus the reduced exceptional divisor of \(f\) where \(B^\sim\) is the birational transform of \(B\). We can write

\[JI(K_W + B_W) = JI f^*(K_X + B) + E'\]

where \(E' \geq 0\) is exceptional/\(X\). It is enough to construct a good log minimal model for \((W/Z, B_W)\) \([1, \text{Remark 2.4}]\). We will show that \((W/Z, B_W)\) also satisfies the finite generation assumptions. Pick any \(\alpha \in H^0(W, mJI(K_W + B_W))\) and let

\[P := (\alpha) + mJI(K_W + B_W) = (\alpha) + mF + mE + mE'\]
Since $P - mE' \equiv 0/X$ and $f_*(P - mE') \geq 0$, we have $P - mE' \geq 0$ by the negativity lemma. Moreover, $\text{Fix}(P - mE') = mE$ hence $P - mE' \geq mE$. This implies that

$$\text{Fix} mJ\ell(K_W + B_W) = mE + mE'$$

Therefore, $R(L_W) \simeq R(L)$ where $L_W = I(K_W + B_W)$. On the other hand, if $G$ is a very ample/Z divisor on $W$, then there is a very ample/Z divisor $G'$ on $X$ such that $G \leq f^*G'$ hence $M^p_G(L_W)$ is a finitely generated $R(L_W)$-module as it is a submodule of $M^p_{f^*G'}(L)$.

Therefore, by replacing $(X/Z, B)$ with $(W/Z, B_W)$ from now on we can assume that $W = X$ and that $f$ is the identity.

Let $g: X \to T$ be the contraction/Z defined by $|F|$. Let $F'$ be a general element of $|rF|$ for some $r \in \mathbb{N}$. We can choose a very ample/Z divisor $G \geq 0$ so that $K_X + B + F' + G$ is nef/Z and that $(X/Z, B + F' + G)$ is dlt. Run the LMMP/Z on $K_X + B + F'$ with scaling of $G$. By boundedness of the length of extremal rays due to Kawamata, if $r$ is sufficiently large, then the LMMP is over $T$, i.e. only extremal rays over $T$ are contracted. Suppose that, perhaps after some log flips and divisorial contractions, we get an infinite sequence of log flips, that is, $K_{X_i} + B_i + F'_i + \lambda_iG_i$ is nef/Z and numerically trivial over $Z_i$ where $B_i, F'_i, G_i$ are birational transforms on $X_i$. By [2], $\lambda := \lim \lambda_i = 0$. Moreover, by the base point free theorem, each $K_{X_i} + B_i + F'_i + \lambda_iG_i$ is semi-ample/Z (of course $K_{X_i} + B_i + F'_i + \lambda_iG_i$ may not be klt but we can use the ampleness of $G$ to reduce the claim to the klt case).

Thus, if $S$ is a component of $E$ not contracted by the LMMP, then there exist

$$0 \leq N_i \sim_\mathbb{Q} K_X + B + F' + \lambda_iG$$

not containing $S$. This contradicts Lemma 3.3 in view of Theorem 3.4 below. Therefore, $E$ is contracted by the LMMP and $K_{X_i} + B_i + F'_i$ is $\mathbb{Q}$-linearly a multiple of $F'_i$.

But $|F'_i|$ is base point free as the LMMP we ran is over $T$. Thus, the LMMP terminates with a good log minimal model.

The following theorem was proved by Nakayama [6, Theorem 6.1.3]. He treated the case $Z = \text{pt}$ but his proof works for general $Z$. For convenience of the reader we present his proof.

**Theorem 3.4.** Assume that $W \to Z = \text{Spec } A$ is a projective morphism from a smooth variety, $w \in W$ a closed point, and $D$ a Cartier divisor on $W$. Assume further that for some effective divisor $C$ there exist an infinite sequence of positive rational numbers $t_1 > t_2 > \cdots$ with $\lim t_i = 0$, and effective $\mathbb{Q}$-divisors $N_i \sim \mathbb{Q} D + t_iC$ with $w \notin \text{Supp } N_i$. Then, there is a very ample/Z divisor $G$ on $W$ such that $w \notin \text{Bs } |mD + G|$ for any $m > 0$.

**Proof.** Let $f: W' \to W$ be the blow up at $w$ with $E$ the exceptional divisor, $D' = f^*D$, $C' = f^*C$, and $N'_i = f^*N_i$. Let $G$ be a very ample/Z divisor on $W$ such that $H := f^*G - K_{W'} - E$ is ample/Z and $H - \epsilon C'$ is also ample/Z for some $\epsilon > 0$. Put $G' = f^*G$. For each $m > 0$, we can write

$$mD' + G' = K_{W'} + E + H + mD' = K_{W'} + E + H - m\epsilon C' + m(t_iC' + D') \sim_\mathbb{Q} K_{W'} + E + H - m\epsilon C' + mN'_i$$
where we choose $t_i$ so that $mt_i < \epsilon$. By assumptions, $E$ does not intersect $N'_i$. Thus, the multiplier ideal sheaf $\mathcal{I}_i$ of $mN'_i$ is isomorphic to $\mathcal{O}_{W'}$ near $E$. In particular, we have the natural exact sequence

$$0 \to \mathcal{I}_i(mD' + G' - E) \to \mathcal{I}_i(mD' + G') \to \mathcal{O}_E(mD' + G') \to 0$$

from which we derive the exact sequence

$$H^0(W', \mathcal{I}_i(mD' + G')) \to H^0(E, (mD' + G')|_E) \to H^1(W', \mathcal{I}_i(mD' + G' - E)) = 0$$

where the last vanishing follows from Nadel vanishing. On the other hand, $(mD' + G')|_E \sim 0$ hence some section of $\mathcal{I}_i(mD' + G')$ does not vanish on $E$. But

$$\mathcal{I}_i(mD' + G') \subseteq \mathcal{O}_{W'}(mD' + G')$$

so some section of $\mathcal{O}_{W'}(mD' + G')$ does not vanish on $E$ which simply means that $w$ is not in $\text{Bs} |mD + G|$. □

**References**


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