Global Torelli theorem for hyperkähler manifolds

Misha Verbitsky

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Hyperkähler manifolds

**DEFINITION:** A hyperkähler structure on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**REMARK:** A hyperkähler manifold has three symplectic forms $\omega_I := g(I \cdot, \cdot)$, $\omega_J := g(J \cdot, \cdot)$, $\omega_K := g(K \cdot, \cdot)$.

**REMARK:** This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves $I, J, K$.

**DEFINITION:** Let $M$ be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called the holonomy group of $M$.

**REMARK:** A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving $I, J, K$).
Holomorphically symplectic manifolds

**DEFINITION:** A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic (2,0)-form.

**REMARK:** Hyperkähler manifolds are holomorphically symplectic. Indeed, \( \Omega := \omega_J + \sqrt{-1} \omega_K \) is a holomorphic symplectic form on \((M, I)\).

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**DEFINITION:** For the rest of this talk, a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.

**DEFINITION:** A hyperkähler manifold \( M \) is called **simple** if \( \pi_1(M) = 0 \), \( H^{2,0}(M) = \mathbb{C} \).

**Bogomolov’s decomposition:** Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

**Further on, all hyperkähler manifolds are assumed to be simple.**
EXAMPLES.

EXAMPLE: An even-dimensional complex vector space.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: A non-compact example: $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ is a resolution of a singularity $\mathbb{C}^2/\pm 1$.

EXAMPLE: Take a 2-dimensional complex torus $T$, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $T/\pm 1$ is called a Kummer surface. It is holomorphically symplectic.

REMARK: Take a symmetric square $\text{Sym}^2 T$, with a natural action of $T$, and let $T[2]$ be a blow-up of a singular divisor. Then $T[2]$ is naturally isomorphic to the Kummer surface $T/\pm 1$.

DEFINITION: A complex surface is called K3 surface if it a deformation of the Kummer surface.

THEOREM: (a special case of Enriques-Kodaira classification) Let $M$ be a compact complex surface which is hyperkähler. Then $M$ is either a torus or a K3 surface.
Hilbert schemes

**DEFINITION:** A Hilbert scheme $M^{[n]}$ of a complex surface $M$ is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension $n$ over $\mathbb{C}$.

**REMARK:** A Hilbert scheme is obtained as a resolution of singularities of the symmetric power $\text{Sym}^n M$.

**THEOREM:** (Fujiki, Beauville) A Hilbert scheme of a hyperkähler surface is hyperkähler.

**EXAMPLE:** A Hilbert scheme of K3 is hyperkähler.

**EXAMPLE:** Let $T$ be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, a universal covering of $T^{[n]}/T$ is called a generalized Kummer variety.

**REMARK:** There are 2 more “sporadic” examples of compact hyperkähler manifolds, constructed by K. O'Grady. All known compact hyperkähler manifolds are these 2 and the three series: tori, Hilbert schemes of K3, and generalized Kummer.
The Teichmüller space and the mapping class group

**Definition:** Let $M$ be a compact complex manifold, and $\text{Diff}_0(M)$ a connected component of its diffeomorphism group (the group of isotopies). Denote by $\tilde{\text{Teich}}$ the space of complex structures on $M$, and let $\text{Teich} := \tilde{\text{Teich}}/\text{Diff}_0(M)$. We call it the Teichmüller space.

**Remark:** Teich is a finite-dimensional complex space (Kodaira-Spencer-Kuranishi-Douady), but often non-Hausdorff.

**Definition:** Let $\text{Diff}_+(M)$ be the group of oriented diffeomorphisms of $M$. We call $\Gamma := \text{Diff}_+(M)/\text{Diff}_0(M)$ the mapping class group. The coarse moduli space of complex structures on $M$ is a connected component of $\text{Teich}/\Gamma$.

**Remark:** This terminology is standard for curves.

**REMARK:** For hyperkähler manifolds, it is convenient to take for Teich the space of all complex structures of hyperkähler type, that is, holomorphically symplectic and Kähler. It is open in the usual Teichmüller space.

**REMARK:** To describe the moduli space, we shall compute Teich and $\Gamma$. 
The Bogomolov-Beauville-Fujiki form

**Theorem**: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where $M$ is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta, \eta)^n$, for some primitive integer quadratic form $q$ on $H^2(M, \mathbb{Z})$, and $c > 0$ an integer number.

**Definition**: This form is called **Bogomolov-Beauville-Fujiki form**. It is defined by the Fujiki’s relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta, \eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \bar{\Omega}^{n-1} - \frac{n-1}{n} \left( \int_X \eta \wedge \Omega^{n-1} \wedge \bar{\Omega}^n \right) \left( \int_X \eta \wedge \Omega^n \wedge \bar{\Omega}^{n-1} \right)$$

where $\Omega$ is the holomorphic symplectic form, and $\lambda > 0$.

**Remark**: $q$ has signature $(b_2 - 3, 3)$. It is negative definite on primitive forms, and positive definite on $\langle \Omega, \bar{\Omega}, \omega \rangle$, where $\omega$ is a Kähler form.
Autonomous of cohomology.

**THEOREM:** Let $M$ be a simple hyperkähler manifold, and $G \subset GL(H^*(M))$ a group of automorphisms of its cohomology algebra preserving the Pontryagin classes. Then $G$ acts on $H^2(M)$ preserving the BBF form. Moreover, the map $G \rightarrow O(H^2(M, \mathbb{R}), q)$ is surjective on a connected component, and has compact kernel.

**Proof.** 

**Step 1:** Fujiki formula $v^{2n} = q(v, v)^n$ implies that $\Gamma_0$ preserves the Bogomolov-Beauville-Fujiki up to a sign. The sign is fixed, if $n$ is odd.

**Step 2:** For even $n$, the sign is also fixed. Indeed, $G$ preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2} \wedge p_1(M) = q(v, v)^{n-1}c$, for some $c \in \mathbb{R}$. The constant $c$ is positive, because the degree of $c_2(B)$ is positive for any Yang-Mills bundle with $c_1(B) = 0$.

**Step 3:** $\sigma(H^2(M, \mathbb{R}), q)$ acts on $H^*(M, \mathbb{R})$ by derivations preserving Pontryagin classes (V., 1995). Therefore $\text{Lie}(G)$ surjects to $\sigma(H^2(M, \mathbb{R}), q)$.

**Step 4:** The kernel $K$ of the map $G \rightarrow G|_{H^2(M, \mathbb{R})}$ is compact, because it commutes with the Hodge decomposition and Lefschetz $sl(2)$-action, hence preserves the Riemann-Hodge form, which is positive definite. ■
Computation of the mapping class group

**Theorem:** (Sullivan) Let $M$ be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geq 3$. Denote by $\Gamma_0$ the group of automorphisms of an algebra $H^*(M, \mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then the natural map $\text{Diff}^+(M)/\text{Diff}_0 \rightarrow \Gamma_0$ has finite kernel, and its image has finite index in $\Gamma_0$.

**Theorem:** Let $M$ be a simple hyperkähler manifold, and $\Gamma_0$ as above. Then

(i) $\Gamma_0 \big|_{H^2(M, \mathbb{Z})}$ is a finite index subgroup of $O(H^2(M, \mathbb{Z}), q)$.

(ii) The map $\Gamma_0 \rightarrow O(H^2(M, \mathbb{Z}), q)$ has finite kernel.

**Proof:** Follows from Sullivan and a computation of $\text{Aut}(H^*(M, \mathbb{R}))$ done earlier. □

**Remark:** (Kollar-Matsusaka, Huybrechts) There are only finitely many connected components of Teich.

**Remark:** The mapping class group acts on the set of connected components of Teich.

**Corollary:** Let $\Gamma_I$ be the group of elements of mapping class group preserving a connected component of Teichmüller space containing $I \in \text{Teich}$. Then $\Gamma_I$ is also arithmetic. Indeed, it has finite index in $\Gamma$. 9
Deformations of holomorphically symplectic manifolds.

**THEOREM:** (Kodaira) A small deformation of a compact Kähler manifold is again Kähler.

**COROLLARY:** A small deformation of a holomorphically symplectic Kähler manifold $M$ is again holomorphically symplectic.

**Proof:** A small deformation $M'$ of $M$ would satisfy $H^{2,0}(M') = H^{2,0}(M)$, however, a small deformation of a non-degenerate $(2,0)$-form remains non-degenerate. ■

**COROLLARY:** Small deformations of hyperkähler manifolds are hyperkähler.

**REMARK:** By the moduli of hyperkähler manifolds we shall understand the deformation space of complex manifolds admitting holomorphically symplectic and Kähler structure.
The period map

Remark: For any $J \in \text{Teich}$, $(M, J)$ is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $P : \text{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map $J$ to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $P : \text{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called the period map.

Remark: $P$ maps Teich into an open subset of a quadric, defined by

$$\text{Per} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}.$$  

It is called the period space of $M$.

Remark: $\text{Per} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$

Theorem: Let $M$ be a simple hyperkähler manifold, and Teich its Teichmüller space. Then

(i) (Bogomolov) The period map $P : \text{Teich} \longrightarrow \text{Per}$ is etale.

(ii) (Huybrechts) It is surjective.

Remark: Bogomolov’s theorem implies that Teich is smooth. It is non-Hausdorff even in the simplest examples.
Hausdorff reduction

**REMARK:** A non-Hausdorff manifold is a topological space locally diffeomorphic to $\mathbb{R}^n$.

**DEFINITION:** Let $M$ be a topological space. We say that $x, y \in M$ are non-separable (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

**THEOREM:** (D. Huybrechts) If $I_1, I_2 \in \text{Teich}$ are non-separable points, then $P(I_1) = P(I_2)$, and $(M, I_1)$ is birationally equivalent to $(M, I_2)$.

**DEFINITION:** Let $M$ be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called a **Hausdorff reduction** of $M$.

**Problems:**
1. $\sim$ is not always an equivalence relation.
2. Even if $\sim$ is equivalence, the $M/\sim$ is not always Hausdorff.

**REMARK:** A quotient $M/\sim$ is Hausdorff, if $M \rightarrow M/\sim$ is open, and the graph $\Gamma_{\sim} \subset M \times M$ is closed.
Weakly Hausdorff manifolds

**DEFINITION:** A point $x \in X$ is called **Hausdorff** if $x \not\sim y$ for any $y \neq x$.

**DEFINITION:** Let $M$ be an $n$-dimensional real analytic manifold, not necessarily Hausdorff. Suppose that the set $Z \subset M$ of non-Hausdorff points is contained in a countable union of real analytic subvarieties of codim $\geq 2$. Suppose, moreover, that

(S) For every $x \in M$, there is a closed neighbourhood $B \subset M$ of $x$ and a continuous surjective map $\Psi : B \to \mathbb{R}^n$ to a closed ball in $\mathbb{R}^n$, inducing a homeomorphism on an open neighbourhood of $x$.

Then $M$ is called a **weakly Hausdorff manifold**.

**REMARK:** The period map satisfies (S). Also, the non-Hausdorff points of Teich are contained in a countable union of divisors.

**THEOREM:** A weakly Hausdorff manifold $X$ admits a Hausdorff reduction. In other words, the quotient $X/\sim$ is a Hausdorff. Moreover, $X \to X/\sim$ is locally a homeomorphism.

This theorem is proven using 1920-ies style point-set topology.
Birational Teichmüller moduli space

**DEFINITION:** The space $\text{Teich}_b := \text{Teich} / \sim$ is called the **birational Teichmüller space** of $M$.

**THEOREM:** The period map $\text{Teich}_b \xrightarrow{\text{Per}} \text{Per}$ is an isomorphism, for each connected component of $\text{Teich}_b$.

The proof is based on two results.

**PROPOSITION:** (The Covering Criterion) Let $X \xrightarrow{\varphi} Y$ be an etale map of smooth manifolds. Suppose that each $y \in Y$ has a neighbourhood $B \ni y$ diffeomorphic to a closed ball, such that for each connected component $B' \subset \varphi^{-1}(B)$, $B'$ projects to $B$ surjectively. Then $\varphi$ is a covering.

**PROPOSITION:** The period map satisfies the conditions of the Covering Criterion.
Global Torelli theorem

DEFINITION: Let $M$ be a hyperkähler manifold, Teich$_b$ its birational Teichmüller space, and $\Gamma$ the mapping class group. The quotient Teich$_b$/\Gamma is called the birational moduli space of $M$.

REMARK: The birational moduli space is obtained from the usual moduli space by gluing some (but not all) non-separable points. It is still non-Hausdorff.

THEOREM: Let $(M,I)$ be a hyperkähler manifold, and $W$ a connected component of its birational moduli space. Then $W$ is isomorphic to $\text{Per}/\Gamma_I$, where $\text{Per} = SO(b_2 - 3,3)/SO(2) \times SO(b_2 - 3,1)$ and $\Gamma_I$ is an arithmetic group in $O(H^2(M,\mathbb{R}), q)$.

A CAUTION: Usually “the global Torelli theorem” is understood as a theorem about Hodge structures. For K3 surfaces, the Hodge structure on $H^2(M,\mathbb{Z})$ determines the complex structure. For $\dim_{\mathbb{C}}M > 2$, it is false.
The birational Hodge-theoretic Torelli theorem

**DEFINITION:** The birational Hodge-theoretic Torelli theorem is true for $M$ if $\Gamma_I$ (the stabilizer of a Torelli component in the mapping class group) is isomorphic to $O^+(H^2(M,\mathbb{Z}),q)$.

**REMARK:** If a birational Hodge-theoretic Torelli theorem holds for $M$, then any deformation of $M$ is up to a bimeromorphic equivalence determined by the Hodge structure on $H^2(M)$.

**THEOREM:** (Markman) The for $M = K3^n$, the group $\Gamma_I$ is a subgroup of $O^+(H^2(M,\mathbb{Z}),q)$ generated by oriented reflections.

**THEOREM:** Let $M = K3^{n+1}$ with $n$ a prime power. Then the (usual) global Torelli theorem holds birationally: two deformations of a Hilbert scheme with isomorphic Hodge structures are bimeromorphic. For other $n$, it is false (Markman).