Seshadri constants via toric degenerations

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Introduction

1. Introduction
2. Toric case
3. Non-toric case
4. Multi-point case
Let $L$ be an ample line bundle on a projective variety $X$ over $\mathbb{C}$. 
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**How can we measure the positivity of $L$?**

- The volume $L^n$ is one basic measure, where $n = \dim X$.
- But it is not enough.
Example

\( X = \mathbb{P}^1 \times \mathbb{P}^1, L_1 = \mathcal{O}(k, k), L_2 = \mathcal{O}(1, k^2). \)
Example

\[ X = \mathbb{P}^1 \times \mathbb{P}^1, \quad L_1 = O(k, k), \quad L_2 = O(1, k^2). \]

Then

- \[ K_X + L_1 \] is nef for \( k \geq 2 \)
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Then

- \( K_X + L_1 \) is nef for \( k \geq 2 \) and very ample for \( k \geq 3 \),

- but \( K_X + L_2 \) is not effective for any \( k > 0 \).
Definition (Demailly ’92)

The Seshadri constant $\varepsilon(X, L; p)$ for $p \in X$ is;
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The Seshadri constant $\varepsilon(X, L; p)$ for $p \in X$ is;

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\varepsilon(X, L; p) := \inf_C \frac{C.L}{\text{mult}_p(C)} > 0
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\mu : \tilde{X} \to X, E = \mu^{-1}(p)
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For a line bundle $L$,

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L \text{ is ample } \Leftrightarrow \inf_{p,C} \frac{C.L}{\text{mult}_p(C)} > 0
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Example

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- $\varepsilon(\mathbb{P}^n, O(1); p) = 1$ for $\forall p,$
- $\varepsilon(\mathbb{P}^1 \times \mathbb{P}^1; O(a, b); p) = \min\{a, b\}$ for $a, b > 0, \forall p,$
Example

- \( \varepsilon(\mathbb{P}^n, O(1); p) = 1 \) for \( \forall p \),
- \( \varepsilon(\mathbb{P}^1 \times \mathbb{P}^1; O(a, b); p) = \min\{a, b\} \) for \( a, b > 0, \forall p \),
- For a smooth cubic surface \( S \subset \mathbb{P}^3 \),

\[
\varepsilon(S, O(1); p) = \begin{cases} 
1 & \text{if } p \in \text{line} \\
3/2 & \text{otherwise.}
\end{cases}
\]
Remark

(1) For $p \geq 8 \in \mathbb{Z}$, $(X; L; p) \Rightarrow \dim \mathbb{Z} \sqrt{\mathbb{Z}}: L \dim \mathbb{Z}$, $\mult p (\mathbb{Z}) \Rightarrow \text{dim}$. In particular, $(X; L; p) \Rightarrow n p L \Rightarrow n$ holds.

(2) For a flat family $(X_t; L_t; p_t)_{t \in T}$ over smooth $T$ and $0 \Rightarrow T$, $(X_t; L_t; p_t) \Rightarrow (X_0; L_0; p_0)$ holds for very general $t$ (lower semicontinuity).
Remark

(1) For $p \in^Y Z \subset X$, 

\[ \dim Z \geq \dim (X; L; p) \] 

In particular, 

\[ (X; L; p) \]

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(1) For $p \in \mathcal{Z} \subset X$,

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(2) For a flat family $(X_t, L_t, p_t)_{t \in T}$ over smooth $T$ and $0 \in T$,

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- Gromov width (Mcduff-Polterovich), and so on.
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- \( \varepsilon(X, L; 1) \geq \frac{1}{\text{dim } X} \) holds (Ein-Küchle-Lazarsfeld),
- abelian varieties, (Nakamaye,Lazarsfeld,etc.,.).
But it is very difficult to compute Seshadri constants in general.
In higher dimensional cases, the following results are known:

- $\varepsilon(X, L; 1) \geq 1/\dim X$ holds (Ein-Küchle-Lazarsfeld),
- abelian varieties, (Nakamaye, Lazarsfeld, etc.,).
- $X$: toric, $p$: torus invariant point (Di Rocco).
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- estimate \( \varepsilon(X, L; 1) \) for toric \( X \),
- find "good" toric degenerations and use lower semicontinuities.
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- estimate $\varepsilon(X, L; 1)$ for toric $X$,
- find ”good” toric degenerations and use lower semicontinuities.

By this strategy, we obtain the following results;
Theorem (Hypersurfaces)

$X \subset \mathbb{P}^{n+1}$: a very general hypersurface of degree $d$. 

Note that the upper bound comes from $\left( X; \mathcal{O}(1); 1 \right)$.
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\( X \subset \mathbb{P}^{n+1} : \text{a very general hypersurface of degree } d. \text{Then it holds that} \)

\[ \lfloor \sqrt[n]{d} \rfloor \leq c(X, O(1); 1) \leq \sqrt[n]{d}. \]
Theorem (Hypersurfaces)

$X \subset \mathbb{P}^{n+1}$: a very general hypersurface of degree $d$. Then it holds that

$$\lfloor \sqrt[n]{d} \rfloor \leq \varepsilon(X, \mathcal{O}(1); 1) \leq \sqrt[n]{d}.$$  

Remark

Note that the upper bound comes from $\varepsilon(X, \mathcal{O}(1); 1) \leq \sqrt[n]{\mathcal{O}(1)^n}$. 


Theorem (Fano 3-folds with Picard number 1)

For each family of smooth Fano 3-folds with Picard number 1 (note that there are 17 such families),
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For each family of smooth Fano 3-folds with Picard number 1 (note that there are 17 such families),

\[ \varepsilon(X, -K_X; 1) = \begin{cases} 
6/5 & (6) \subset \mathbb{P}(1, 1, 1, 1, 3) \\
4/3 & (4) \subset \mathbb{P}^4 \\
3/2 & (2) \cap (3) \subset \mathbb{P}^5 \\
2 & otherwise \\
3 & (2) \subset \mathbb{P}^4 \\
4 & \mathbb{P}^3 
\end{cases} \]

holds, where \( X \) is a very general member in the family.
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The polarized toric variety corresponding to \( P \) is

\[ (X_P, L_P) := (\text{Proj} \bigoplus_{k \geq 0} V_{kP}, \mathcal{O}(1)), \]

where \( V_{kP} := \bigoplus_{u \in kP \cap M} \mathbb{C}x^u \subset \mathbb{C}[M]. \]
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\[ P \subset M_{\mathbb{R}} : \text{integral polytope of dim } n \]
\[ v < P : \text{a vertex, } x_v \subset X_P : \text{the torus invariant point} \]
Then \( \varepsilon(X_P, L_P; x_v) = \min\{ |\tau| \mid v < \tau < P, \dim \tau = 1 \} \) holds.
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For \( \sigma < P \) and \( p \in O_\sigma \),
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Remark

For \( \sigma < P \) and \( p \in O_{\sigma} \), it holds that

\[
\varepsilon(X_P, L_P; p) = \min \{\varepsilon(X_{\sigma}, L_{\sigma}; p), \varepsilon(X_{P'}, L_{P'}; x_{v'})\},
\]

where \( \pi : M_{\mathbb{R}} \to M_{\mathbb{R}}/(\mathbb{R}(\sigma - \sigma)) \) and
\( P' = \pi(P), v' = \pi(\sigma) \).
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\[ \pi : M_R \rightarrow M'_R : \text{lattice projection with rank } M = n, \]
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For \( u' \in \pi(P) \cap M'_\mathbb{Q} \), set \( P(u') = \pi^{-1}(u') \cap P \).
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\[
\min \{ \varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}), \varepsilon(X_{P(u')}, L_{P(u')}; 1_{P(u')}) \} \\
\leq \varepsilon(X_P, L_P; 1_P) \leq \varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)})
\]
We can construct a rational map \( \phi \): \( X \to \mathbb{P}^d \) such that \( X(\mathbb{P}) \); the general fiber of \( \phi \). We study \( C \): \( L \mathbb{P} = \text{mult}_1 \mathbb{P}(C) \) in case of \( \phi(C) = \text{pt} \); or, \text{pt} separably.
Idea of proof.

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then \( \varepsilon(X_P, L_P; 1_P) = \varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}) \).
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$$= \min\{1, \varepsilon(X_{P_{n-1}}, L_{P_{n-1}}; 1_{P_{n-1}})\}$$

$$\leq \varepsilon(X_{P_n}, L_{P_n}; 1_{P_n}) \leq \varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}) = 1$$
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$$\leq \varepsilon(X_{P_n}, L_{P_n}; 1_{P_n}) \leq \varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}) = 1$$

Inductively, we have $\varepsilon(X_{P_n}, L_{P_n}; 1_{P_n}) = 1$. 
Example

(1) $P = P_n := \text{conv}(0, e_1, \ldots, e_n) \subset \mathbb{R}^n$, 
\pi : \mathbb{R}^n \rightarrow \mathbb{R} : n$-th projection.
Since $\pi(P) = [0, 1]$, $P(0) = P_{n-1}$, it holds that

$$\min\{\varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}), \varepsilon(X_{P(0)}, L_{P(0)}; 1_{P(0)})\}$$
$$= \min\{1, \varepsilon(X_{P_{n-1}}, L_{P_{n-1}}; 1_{P_{n-1}})\}$$
$$\leq \varepsilon(X_{P_n}, L_{P_n}; 1_{P_n}) \leq \varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}) = 1$$

Inductively, we have $\varepsilon(X_{P_n}, L_{P_n}; 1_{P_n}) = 1$. 
Note that $(X_{P_n}, L_{P_n}) = (\mathbb{P}^n, \mathcal{O}(1))$. 
Example

(2) $P = \text{conv}(e_1, e_2, -e_1 - e_2)$,

$\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$: 2-nd projection.

Then we have

$\min\{2, 3/2\} = 3/2 \leq \varepsilon(X_P, L_P; 1_P) \leq 2$.

Note that $X_P$ is the cubic surface in $\mathbb{P}^3$ defined by

$T_0^3 = T_1 T_2 T_3$, 

with $T_i$ being the monomials of degree $i$.
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\[
\varepsilon(X_P, L_P; 1_P) \geq \min_{1 \leq i \leq n} \frac{a_i + \cdots + a_n + 1}{a_{i+1} + \cdots + a_n + 1}.
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In (2) $a_1 = a_2 = 1$, hence

$$\min \left\{ \frac{a_1 + a_2 + 1}{a_2 + 1}, \frac{a_2 + a_1}{1} \right\} = \min \left\{ \frac{3}{2}, 2 \right\} = \frac{3}{2}.$$
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Assume that \((X_0, L_0)^{nor} \cong (X_P, L_P)\) for some \(P \subset M_\mathbb{R}.\)

Then

\[\varepsilon(X_t, L_t; 1) \geq \varepsilon(X_P, L_P; 1_P)\]

holds for very general \(t \in T.\)
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In particular, we have

\[ \lfloor \sqrt[n]{d} \rfloor \leq \varepsilon(X, O(1); 1) \leq n\sqrt[2]{d}. \]
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$X$ degenerates to

$X_0 := (T_0^d = T_1^{c_1-c_2} \cdots T_n^{c_n-c_{n+1}} T_{n+1}^{c_{n+1}}) \subset \mathbb{P}^{n+1}.$
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The last part follows if we take

$c_n = c, c_{n-1} = c^2, \ldots, c_2 = c^{n-1}$ for $c = \lfloor \sqrt[d]{n} \rfloor$. $\square$
Example

(1) When $n = 2$, 
$$(X^2d; O(1); 1)_{\min f\lceil p_d \rceil; d} = \lceil p_d \rceil g = d = \lceil p_d \rceil$$

follows from 
$$1 \lceil p_d \rceil.$$ 
Thus we have 
$$(X^2d; O(1); 1)_{\max f\lfloor p_d \rfloor; d} = \lfloor p_d \rfloor g.$$ 

For example, 
$$(X^27; O(1); 1)_{\max f2; 7} = 3 = \lceil p_d \rceil 7 = 3.$$ 

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(3) $\varepsilon(X_c^n, O(1); 1) = c$ holds for any $c, n \in \mathbb{N}$. 
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Then it holds that $\varepsilon(X, O(1); 1) = d_1/(d_1 - 1)$. 

Proof.

We prove only $k = 1$ case, thus we show $\varepsilon(X, O(1); 1) = (n + 1)/n$ since $d_1 = n + 1$. 

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Thus $\varepsilon(X, O(1); 1) \leq C \cdot O(1)/ \text{mult}_p(C) = (n + 1)/n$. 

$\square$
Theorem (Fano 3-folds with Picard number 1)

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\[ \varepsilon(X, -K_X; 1) = \begin{cases} 
6/5 & (6) \subset \mathbb{P}(1, 1, 1, 1, 3) \\
4/3 & (4) \subset \mathbb{P}^4 \\
3/2 & (2) \cap (3) \subset \mathbb{P}^5 \\
2 & \text{otherwise} \\
3 & (2) \subset \mathbb{P}^4 \\
4 & \mathbb{P}^3 
\end{cases} \]

holds, where \( X \) is a very general member in the family.
Ilten, Lewis, and Przyjalkowski showed that such $X$ degenerates to a toric variety. We use it to show $\geq$. $\leq$ is proved by finding a suitable curve $C \subset X$. \qed
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Which polarized variety degenerates to a polarized variety whose normalization is toric?
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**Question**

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Anderson gave an interesting partial answer;
Example

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Thus \(\varepsilon(X, L; 1) \geq \varepsilon(X_{\Delta(L)}, L_{\Delta(L)}; 1_{\Delta(L)})\) holds in this case.
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I proved that $\varepsilon(X, L; 1) \geq \varepsilon(X_{\Delta(L)}, L_{\Delta(L)}; 1_{\Delta(L)})$ holds without the finitely generatedness condition if we define $\varepsilon(X_\Delta, L_\Delta; 1_\Delta)$ for any closed convex set $\Delta \subset \mathbb{R}^n$ suitably.
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for very general $p_1, \ldots, p_r \in X$.

Remark

$$
\varepsilon(X, L; t\overline{m}) = t^{-1} \varepsilon(X, L; \overline{m}) \text{ holds for any } t > 0.
$$
Proposition

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Proposition

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Assume that general fibers are red. and irreducible. and \(X_0 = \bigcup_{i=1}^{r} Y_i : \text{reduced.}\)

Then

\[\varepsilon(X_t, L_t; \varepsilon_1, \ldots, \varepsilon_r) \geq 1\]

holds for very general \(t \in T,\)

where \(\varepsilon_i = \varepsilon(Y_i, L_0|_{Y_i}; 1).\)
Theorem

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\[ \left\lfloor \sqrt[n]{d / \sum_{i=1}^{r} m_i^n} \right\rfloor \leq \varepsilon(X, O(1); \overline{m}) \leq \sqrt[n]{d / \sum_{i=1}^{r} m_i^n} \]

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**Remark**

Note that the above theorem is false for \( \overline{m} \in (\mathbb{R}_{>0})^r \) in general.
Sketch of proof.

Let $d_1, \ldots, d_r \in \mathbb{N} \setminus 0$ such that $\sum d_i = d$. 
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Let $d_1, \ldots, d_r \in \mathbb{N} \setminus \{0\}$ such that $\sum d_i = d$. Since $X_d$ degenerates to $\bigcup_{i=1}^r X_{d_i}$, we have $\varepsilon(X_d, O(1); \varepsilon_1, \ldots, \varepsilon_r) \geq 1$ for $\varepsilon_i := \varepsilon(X_{d_i}, O(1); 1)$. 


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where $c = \lfloor \sqrt[n]{d / \sum_{i=1}^r m_i^n} \rfloor$. 
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Then $\varepsilon_i = \varepsilon(X_{d_i}, \mathcal{O}(1); 1) \geq \lfloor \sqrt[n]{d_i} \rfloor \geq cm_i$. 

\[\square\]