Characterization of varieties of Fano type via singularities of Cox rings II

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Schedule

1. Cox ring and Mori dream space
2. $D$-minimal model program
3. Review of Okawa’s talk
4. MDS of gl. $F$-reg. type is of Fano type
5. Characterization of varieties of Fano type via singularities of Cox rings
6. Proof of Key lemma
7. Case of log Calabi–Yau
8. Application
9. Open question
Definition(Cox ring)

$X$: a normal $\mathbb{Q}$-fac. proj. var. $/ k$
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$X$: a normal $\mathbb{Q}$-fac. proj. var. / $k$ s.t. $\text{Pic}(X)_\mathbb{Q} \cong N^1(X)_\mathbb{Q}$. 
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$\Gamma \subset \text{Div}(X)$: f.g. group of Cartier div. on $X$ s.t.

$$\Gamma_\mathbb{Q} \to \text{Pic}(X)_\mathbb{Q}; \ D \mapsto \mathcal{O}_X(D)$$

is iso.
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$\Gamma \subset \text{Div}(X)$: f.g. group of Cartier div. on $X$ s.t.

$$\Gamma_\mathbb{Q} \rightarrow \text{Pic}(X)_\mathbb{Q}; \ D \mapsto O_X(D)$$

is iso. The multi-sec. ring

$$R_X(\Gamma) = \bigoplus_{D \in \Gamma} H^0(X, O_X(D))$$

is called a Cox ring of $X$. 
Definition [Mori dream space]

$X$: normal proj. var. / $k$.

$X$: Mori dream space (or MDS for short),

(i) $X$: $\mathbb{Q}$-fac. & Pic($X$) $\mathbb{Q} \cong N_1(X)$,

(ii) $\mathrm{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles,

(iii) $\exists$ a finite collection of small bir. maps $f_i: X \to X_i$ s.t. $X_i$ satisfies (i) & (ii), and $\mathrm{Mov}(X) = \bigcup_i f_i(\mathrm{Nef}(X_i))$.

Theorem [Hu–Keel].

$X$: $\mathbb{Q}$-fac. normal proj. var. / $k$ such that Pic($X$) $\mathbb{Q} \cong N_1(X)$.

$X$: MDS, $\mathrm{Cox}(X)$: f.g. $k$-alg. - Y. Gongyo, S. Okawa, A. Sannai, & S. Takagi

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Character. of Fano type via Cox rings II

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\(X\): normal proj. var. / \(k\).

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(i) \(X\): Q-fac. & \(\text{Pic}(X)_\mathbb{Q} \cong N^1(X)_\mathbb{Q}\),
Definition [Mori dream space]

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[Theorem [Hu–Keel]]
Definition[Mori dream space]

X: normal proj. var. / k.

X: Mori dream space (or MDS for short) ⇔

(i) X: Q-fac. & Pic(X)_Q ≃ N^1(X)_Q,

(ii) Nef(X) is the affine hull of finitely many semi-ample line bundles,

(iii) ∃ a finite collection of small bir. maps f_i : X → X_i s.t. X_i satisfies (i) & (ii), and Mov(X) = ∪_i f_i^*(Nef(X_i)).

Theorem[Hu–Keel]

X: Q-fac. normal proj. var. / k such that Pic(X)_Q ≃ N^1(X)_Q.

X: MDS ⇔ Cox(X) : f.g. k-alg.
Given $X \& D(\sim)$; Ask $D$ is nef or not, $D$ is nef $\Rightarrow$ $X$ is $D$-minimal model, $D$ is not nef $\Rightarrow$ Find a curve $C$ s.t. $D: C < 0 \& R: R = 0 \{ C \} \subset N_1(X) \Rightarrow$ $X$ extremal, $\Rightarrow$ Construct a proj. mor. $\phi: X \to Y$ of conn. fibers s.t. Curve $C'$ s.t. $\phi(C)$ is a point, $\{ C' \} \subset R$, (i) $\phi$: bir. contracts a divisor $\Rightarrow$ Replacing as $X = Y \& D = \phi(D)$, go back $\Rightarrow$, (ii) $\phi$: bir. contracts no divisors, $\Rightarrow$ Constructing $D$-flip $X_d X + Y$. Gongyo, S. Okawa, A. Sannai, & S. Takagi University of Tokyo Algebraic Geometry Conference Chulalongkorn University, Bangkok, Thailand (21 December, 2011 5 / 23)
$D$-MMP

Given $X$ & $D$ – (◊),
**$D$-MMP**

Given $X$ & $D$ – (◇),

Ask $D$ is nef or not,
**D-MMP**

Given $X$ and $D$ - ($\Diamond$),

Ask $D$ is nef or not,

$D$ is nef $\Rightarrow X$: $D$-minimal model,
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Given $X$ & $D$ – (◇),

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Given $X$ & $D - (\Diamond)$,

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$\Rightarrow$ Find a curve $C$ s.t. $D.C < 0$ & $R := \mathbb{R}_{\geq 0}[C] \subset \text{N}^1(X)_{\mathbb{R}}$: extremal,
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Curve $C'$ s.t. $\varphi(C)$ is a point $\iff [C'] \in R$, 
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Given $X$ & $D - (\Diamond)$,
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Given $X$ & $D - (\Diamond)$,

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\text{Curve } C' \text{ s.t. } \varphi(C) \text{ is a point } \iff [C'] \in R,
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Given $X \& D \rightarrow (\Diamond)$,
Ask $D$ is nef or not,

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(i) $\varphi$: bir. contracts a divisor
$\Rightarrow$ Replacing as $X := Y \& D := \varphi_*D$, go back $(\Diamond)$,

(ii) $\varphi$: bir. contracts no divisors,
$\Rightarrow$ Constructing $D$-flip $X \rightarrow X^+$,
**$D$-MMP**

i.e. $\exists$ small bir. mor. $\varphi^+ : X^+ \to Y$ s.t. $X^+: \text{Q-fac.}$, $\rho(X^+ / Y) = 1$, & the str. trans. $D^+: \varphi^+$-ample,
$D$-MMP

i.e. $\exists$ small bir. mor. $\varphi^+ : X^+ \to Y$ s.t. $X^+$: Q-fac., $\rho(X^+/Y) = 1$, & the str. trans. $D^+ : \varphi^+$-ample,
⇒ Replacing as $X := X^+$ & $D := D^+$, go back (◇),
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i.e. \( \exists \) small bir. mor. \( \varphi^+ : X^+ \to Y \) s.t. \( X^+ : \text{Q-fac.} \), \( \rho(X^+ / Y) = 1 \), &

the str. trans. \( D^+ : \varphi^+ \)-ample,

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(iii) \( \varphi \) has a positive dim. general fibers.
$D$-MMP

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$\Rightarrow$ Replacing as $X := X^+$ \& $D := D^+$, go back ($\Diamond$),

(iii) $\varphi$ has a positive dim. general fibers.

$\Rightarrow$ $\varphi : X \to Y$: $D$- Mori fiber space.
D-MMP

i.e. \( \exists \) small bir. mor. \( \varphi^+ : X^+ \to Y \) s.t. \( X^+ : \text{Q-fac.}, \rho(X^+ / Y) = 1, \) & the str. trans. \( D^+ : \varphi^+ \)-ample,
\( \Rightarrow \) Replacing as \( X := X^+ \) & \( D := D^+ \), go back (\( \diamond \)),

(iii) \( \varphi \) has a positive dim. general fibers.
\( \Rightarrow \varphi : X \to Y : D\)-Mori fiber space.

Repeat the process:

\[
X_0 = X \to X_1 \to \cdots \to X_i \cdots
\]
\textbf{D-MMP} \\
\text{i.e. } \exists \text{ small bir. mor. } \varphi^+: X^+ \to Y \text{ s.t. } X^+: \text{Q-fac.}, \rho(X^+/Y) = 1, \& \text{the str. trans. } D^+: \varphi^+\text{-ample},

\Rightarrow \text{Replacing as } X := X^+ \& D := D^+, \text{ go back (◊),}

(iii) \varphi \text{ has a positive dim. general fibers.}

\Rightarrow \varphi : X \to Y: D\text{- Mori fiber space.}

Repeat the process:

\[ X_0 = X \to X_1 \to \cdots \to X_i \cdots . \]

For looking for $D$-minimal models or $D$-Mori fiber spaces, we run the program.
Theorem[Hu–Keel]

\( X: \text{MDS} \) & \( D: \text{divisor on } X. \)

Then \( D \)-MMP run and terminates.

Moreover each \( X_i \) is also a MDS.
Theorem [Hu–Keel]

\textbf{X: MDS} \& \textbf{D: divisor on } X.

Then \textbf{D-MMP} run and terminates.

Moreover each \textbf{X}_i is also a MDS.

Theorem [Hu–Keel]

\textbf{X: MDS}.

Then \exists a finite collection of rat. cont. maps \( f_i : X \rightarrow X_i \) s.t. \( \forall \) rat. cont. map \( g : X \rightarrow Y \), \( \exists i \) s.t. \( f_i \simeq g \).
Review of Okawa’s talk

Strongly $F$-regular
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$k$: $F$-finite field of char. $p > 0.$
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An int. dom. $R$ is f.g. alg./ $k$ is strongly $F$-regular
Review of Okawa’s talk

Strongly $F$-regular

$k$: $F$-finite field of char. $p > 0$.

An int. dom. $R$ is f.g. alg./ $k$ is

$R$: strongly $F$-regular $\iff 0 \neq \forall c \in R \exists e > 0$ s.t.

$$cF^e : R \rightarrow F^e_*R \rightarrow F^e_*R$$

splits as $R$-mod.
Globally $F$-regular

$k$: $F$-finite of char. $p > 0$.

$X$: normal var./ $k$. 
Globally $F$-regular

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$X$: globally $F$-regular
Globally $F$-regular

$k$: $F$-finite of char. $p > 0$.

$X$: normal var./ $k$.

$X$: globally $F$-regular $\iff \forall D \geq 0 \exists e > 0$ s.t. $cF^e : O_X \to F^e_*O_X \to F^e_*(O_X(D)),$

splits as $O_X$-mod. , where $c \in H^0(X, O_X(D))$ s.t. $\text{div}_0(c) = D$. 
Globally $F$-regular

$k$: $F$-finite of char. $p > 0$.

$X$: normal var./ $k$.

$X$: globally $F$-regular $\iff \forall D \geq 0 \exists e > 0$ s.t.

$$cF^e : O_X \to F_*^eO_X \to F_*^e(O_X(D)),$$

splits as $O_X$-mod. , where $c \in H^0(X, O_X(D))$ s.t. $\text{div}_0(c) = D$.

Proposition (K. Smith)

As above,
Globally $F$-regular

$k$: $F$-finite of char. $p > 0$.

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$X$ : globally $F$-regular $\iff \forall D \geq 0 \exists e > 0$ s.t.

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splits as $O_X$-mod., where $c \in H^0(X, O_X(D))$ s.t. $\text{div}_0(c) = D$.

Proposition (K. Smith)

As above,

$X$ is gl. $F$-regular
Globally $F$-regular

$k$: $F$-finite of char. $p > 0$.

$X$: normal var./ $k$.

$X$ : globally $F$-regular $\iff \forall D \geq 0 \exists e > 0 \text{ s.t.}$

$$cF^e : O_X \to F^e_*O_X \to F^e_*(O_X(D)),$$

splits as $O_X$-mod. , where $c \in H^0(X, O_X(D))$ s.t. $\text{div}_0(c) = D$.

Proposition (K. Smith)

As above,

$X$ is gl. $F$-regular $\iff \exists$ ample div. $H$ s.t. the sec. ring $R(X, H)$: strongly $F$-regular.
Theorem P (Hashimoto, Sannai)

$k$: $F$-finite of char. $p > 0$.

$X$: proj. normal var./ $k$. 
Theorem P (Hashimoto, Sannai)

$k$: $F$-finite of char. $p > 0$.

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$X$: gl. $F$-regular.
Theorem P (Hashimoto, Sannai)

$k$: \( F \)-finite of char. \( p > 0 \).

\( X \): proj. normal var./ \( k \).

\( X \): gl. \( F \)-regular.

\( \forall \Gamma \subset \text{Div} (X) \): semi-group of Cartier div.

\[
R_X(\Gamma) = \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D))
\]

is strongly \( F \)-regular.
Theorem F [GOST]

$X$: MDS.
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If $X$ is of gl. $F$-regular type,
Theorem F [GOST]

\[ X: \text{MDS.} \]

If \( X \) is of gl. \( F \)-regular type, then \( X \) is of Fano type, i.e., there exists \( \Delta \geq 0 \) such that \((X, \Delta)\) is klt and \(- (K_X + \Delta)\) is ample.
Theorem F [GOST]

$X$: MDS.

If $X$ is of gl. $F$-regular type, then $X$ is of Fano type, i.e.,

$\exists \Delta \geq 0$ s.t. $(X, \Delta)$ is klt & $-(K_X + \Delta)$: ample.

Proof.
Theorem F [GOST]

$X$: MDS.

If $X$ is of gl. $F$-regular type, then $X$ is of Fano type, i.e.,

$\exists \Delta \geq 0 \text{ s.t. } (X, \Delta) \text{ is klt } \& -(K_X + \Delta): \text{ ample.}$

Proof. Running $(-K_X)$-MMP:
Theorem F [GOST]

\( X \): MDS.

If \( X \) is of gl. \( F \)-regular type, then \( X \) is of Fano type, i.e.,
\[ \exists \Delta \geq 0 \text{ s.t. } (X, \Delta) \text{ is klt & } -(K_X + \Delta) : \text{ ample}. \]

Proof. Running \((-K_X)\)-MMP:

\[ X_0 = X \to X_1 \to \cdots \to X_l, \]
Theorem F [GOST]

$X$: MDS.

If $X$ is of gl. $F$-regular type, then $X$ is of Fano type, i.e.,

$\exists \Delta \geq 0$ s.t. $(X, \Delta)$ is klt & $-(K_X + \Delta)$: ample.

Proof. Running $(-K_X)$-MMP:

$X_0 = X \rightarrow X_1 \rightarrow \cdots \rightarrow X_I$,

where $X_I$: a final model.
Theorem F [GOST]

$X$: MDS.

If $X$ is of gl. $F$-regular type, then $X$ is of Fano type, i.e.,

$\exists \Delta \geq 0$ s.t. $(X, \Delta)$ is klt & $-(K_X + \Delta)$: ample.

Proof. Running $(-K_X)$-MMP:

$$X_0 = X \mapsto X_1 \mapsto \cdots \mapsto X_f,$$

where $X_f$: a final model. We know each $X_i$ is also a MDS of gl. $F$-reg. type since, in general,

any images of ver. of gl. $F$-reg. type are also of gl. $F$-reg. type, and gl. $F$-reg. preserve under isom. in codim 1.
Thus we see:

Claim.

\[ X_l \] is \((K_{X_l})\)-minimal model such that \(K_{X_l}\) is big.

Proof of Claim:

We know \(X_l\); \(p\)-reduction of Fano type under taking mod. \(p\)-reduction from Schwede–Smith’s theorem.

Theorem [Schwede–Smith].

\[ X_p : \text{gl. } F\text{-regular variety over a } F\text{-finite field of characteristic } p > 0. \]

Then \(X_p\) is of Fano type, i.e. \(9 \Delta_p \geq 0 \text{ s.t. } (X_p; \Delta_p) \text{ is klt} \& (K_{X_p} + \Delta_p) \text{ ample.} \)

Remark!

\(\Delta_p\) depends on \(p\). Thus we cannot lift it on \(X\).
Thus we see:

Claim

$X_l$ is $(-K_X)$-minimal model such that $-K_{X_l}$ is big.
Thus we see:

**Claim**

$X_I$ is $(-K_X)$-minimal model such that $-K_{X_I}$ is big.

**Proof of Claim:**
Thus we see:

Claim

$X_l$ is $(-K_X)$-minimal model such that $-K_{X_l}$ is big.

Proof of Claim: We know $X_{l,p}$ is of Fano type under taking mod. $p$-reduction from Schwede–Smith’s theorem:

Theorem [Schwede–Smith]

$X_p : \text{gl. } F\text{-regular variety over a } F\text{-finite field of characteristic } p > 0$.

Then $X_p$ is of Fano type, i.e. $\exists \Delta_p \geq 0$ s.t. $(X_p, \Delta_p)$ is klt & $-(K_{X_p} + \Delta_p)$: ample.
Thus we see:

**Claim**

\( X_l \) is \((-K_X)\)-minimal model such that \(-K_{X_l}\) is big.

**Proof of Claim:** We know \( X_{l,p} \) is of Fano type under taking mod. \( p \)-reduction from Schwede–Smith’s theorem:

**Theorem [Schwede–Smith]**

\( X_p : \text{gl. } F\text{-regular variety over a } F\text{-finite field of characteristic } p > 0. \)

Then \( X_p \) is of Fano type, i.e. \( \exists \Delta_p \geq 0 \text{ s.t. } (X_p, \Delta_p) \text{ is klt & } -(K_{X_p} + \Delta_p) : \text{ample.} \)

**Remark!** \( \Delta_p \) depends on \( p \). Thus we can not lift it on \( X \).
Assume $f : X_l \to Y$ is $(-K_X)$-Mori fiber space.
Assume $f : X_l \to Y$ is $(-K_X)$-Mori fiber space. Let $C$ a $f$-contracting curve.

$$K_{X_l}.C > 0.$$
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$$K_{X_l}.C > 0.$$ 

Taking reduction mod $p$,
Assume $f : X_l \to Y$ is $(-K_X)$-Mori fiber space. Let $C$ a $f$-contracting curve.

$$K_{X_l}.C > 0.$$ 

Taking reduction mod $p$, it holds $K_{X_l,p}.C_p > 0$ and $C_p$ is movable.
Assume $f : X_l \to Y$ is $(-K_X)$-Mori fiber space. Let $C$ a $f$-contracting curve.

$$K_{X_l}.C > 0.$$ 

Taking reduction mod $p$, it holds $K_{X_{l,p}}.C_p > 0$ and $C_p$ is movable. However $\exists \Delta_p \geq 0$ s.t. $-(K_{X_p} + \Delta_p)$: ample.
Assume $f : X_l \to Y$ is $(-K_X)$-Mori fiber space. Let $C$ a $f$-contracting curve.

$$K_{X_l}.C > 0.$$  

Taking reduction mod $p$, it holds $K_{X_l,p}.C_p > 0$ and $C_p$ is movable. However $\exists \Delta_p \geq 0$ s.t. $-(K_{X_p} + \Delta_p)$: ample. Since $\Delta_p.C_p \geq 0$, this is a contradiction!
Assume $f : X_l \to Y$ is $(-K_X)$-Mori fiber space. Let $C$ a $f$-contracting curve. 

$$K_{X_l}.C > 0.$$ 

Taking reduction mod $p$, it holds $K_{X_{l,p}}.C_p > 0$ and $C_p$ is movable. However $\exists \Delta_p \geq 0$ s.t. $-(K_{X_p} + \Delta_p)$: ample. Since $\Delta_p.C_p \geq 0$, this is a contradiction! Thus $-K_{X_l}$ is nef. In particular, $-K_{X_l}$ is semi-ample since $X_l$ is a MDS.
Assume $f : X_l \to Y$ is $(-K_X)$-Mori fiber space. Let $C$ a $f$-contracting curve.

$$K_{X_l}.C > 0.$$ 

Taking reduction mod $p$, it holds $K_{X_l,p}.C_p > 0$ and $C_p$ is movable. However $\exists \Delta_p \geq 0$ s.t. $-(K_{X_p} + \Delta_p)$: ample. Since $\Delta_p.C_p \geq 0$, this is a contradiction!

Thus $-K_{X_l}$ is nef. In particular, $-K_{X_l}$ is semi-ample since $X_l$ is a MDS. Thus we see:

$$(-K_{X_l})^{\dim X} = (-K_{X_l,p})^{\dim X} > 0,$$

since $-K_{X_l,p}$ is nef and big.
Assume \( f : X_l \to Y \) is \((-K_X)\)-Mori fiber space. Let \( C \) a \( f \)-contracting curve.

\[
K_{X_l}.C > 0.
\]

Taking reduction mod \( p \), it holds \( K_{X_{l,p}}.C_p > 0 \) and \( C_p \) is movable. However \( \exists \Delta_p \geq 0 \) s.t. \( -(K_{X_p} + \Delta_p) \): ample.

Since \( \Delta_p.C_p \geq 0 \), this is a contradiction!

Thus \(-K_{X_l} \) is nef. In particular, \(-K_{X_l} \) is semi-ample since \( X_l \) is a MDS. Thus we see:

\[
(-K_{X_l})^{\dim X} = (-K_{X_{l,p}})^{\dim X} > 0,
\]

since \(-K_{X_{l,p}} \) is nef and big. Finish the proof of Claim.
In particular, Hara–Watanabe’s theorem says $X$ has only log terminal singularities.

**Theorem[Hara-Watanabe]**

$X$: $\mathbb{Q}$-Gor. normal var./ $k$ of ch.=0.

If $X$ is of strongly $F$-regular type, then $X$ has only log terminal singularities.
In particular, Hara–Watanabe’s theorem says $X$ has only log terminal singularities.

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Now by induction it suffice to show $X$ is of Fano type under the assumption that $X_1$ is of Fano type.

Assume $\exists \Delta_1 \geq 0$ s.t. $(X_1, \Delta_1)$ is klt & $-(K_{X_1} + \Delta_1)$: ample.
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\( \Rightarrow X \) is of Fano type since \(- (K_X + \Delta) = -g^+*(K_Y + \Delta_Y) \) is nef-big, and klt, where \( \Delta \): the str. trans. of \( \Delta_Y \).
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Finish the proof of Theorem F.
Theorem C [GOST]

Let $X$ be a $\mathbb{Q}$-fac. proj. normal var. over $\mathbb{C}$.

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Remark that we don’t know Cox(\( X \)) \( p \) = Cox(\( X \)).
Character of Fano type via Cox rings

\[(\Rightarrow)\]
$(\Rightarrow)$

Key lemma

$X$: MDS of gl. $F$-reg. type / $C$ & $\Gamma$: f.g. semi-group of Cartier divisors.

Then there exists $m \in \mathbb{N}$ s.t. $R(X, m\Gamma)_p = R(X_p, m\Gamma_p)$ for a mod $p$ reduction.
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Show $\exists m \in \mathbb{N}$ s.t.

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\( \exists \) a finite collection of birat. cont. maps \( f_i : X \rightarrow X_i \) & cont. mor. \( g_{i,j} : X_i \rightarrow Y_{i,j} \) s.t. \( \forall D \in \mathcal{G}, \exists i, j \) s.t. \( f_i : X \rightarrow X_i \) is some \( D \)-MMP with \( g_{i,j} \) \( D \)-can. model or \( D \)-Mori fiber space.
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(i) When \( D \) is not effective, we see

\[
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as a similar arguments to the proof of Claim.
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Thus we see $\exists m \in \mathbb{N}$ s.t.

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Theorem CY [GOST]

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$X$: MDS.

If $X$ is of dense gl. $F$-split type, then $X$ is of Calabi–Yau type, i.e.,

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Character. of Fano type via Cox rings II
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Theorem +[Fujino–Takagi]

$X$: klt Mori dream surface such that $K_X \sim_{\mathbb{Q}} 0$. 

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Character. of Fano type via Cox rings II

21, December, 2011
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Theorem +[Fujino–Takagi]

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Then \( \text{Cox}(X) \) has only lc singularities.
We can give another proof of the following:

Cor. [Shokurov– Prokhorov, Fujino–Gongyo]

\[ f : X \rightarrow Y : \text{surj. prom. mor. of normal proj. var. /} \mathbb{C}. \]

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