Hyperplane Arrangements

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I. Setup

II. Topology

III. Geometry

IV. Arithmetic
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\[ D \text{ is indecomposable if } f \neq g_1(u)g_2(t) \text{ for any choice of coordinates } (u, t) \text{ of } \mathbb{C}^n \text{ and any non-constant polynomials } g_1, g_2. \]
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- the codimensions $\text{codim} : \mathcal{L}(D) \to \mathbb{Z}$;

- the multiplicities $m_i$ of the hyperplanes $D_i$. 
II. Topology

Theorem (Orlik-Solomon 1980):

\[ D = \text{central} \Rightarrow H^*(V, C) = \Lambda^* \left( \sum_{i \in S} C e_i \right) / (\text{de}_I | I \subset S \text{ linearly dependent}) \]

where if \( I = \{i_1, \ldots, i_k\} \) then \( \text{de}_I = \sum_{j=1}^k (-1)^j e_{i_1} \ldots \hat{e}_{i_j} \ldots e_{i_k} \).
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Still unknown even for a cone over a line arrangement in $\mathbb{P}^2$ with at most triple points.
The *resonance varieties*:

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R_{ij}(V) := \{ v \in H_1(V, \mathbb{C}) | \dim H^i(H^\bullet(V, \mathbb{C}), v \cup .) \geq j \}
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are combinatorial invariants (first defined by Falk).

Theorem (Cohen-Orlik):

\[ R_{ij}(V) = T_C V(R_{ij}(V)) \]

Builds on Arapura (Hodge theory), Cohen - Suciu, Libgober - Yuzvinsky.

Corollary:

\[ R_{ij}(V) = \text{union of vector subspaces of } H_1(V, \mathbb{C}) \]
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**Theorem (Cohen-Orlik):** \( R_j^i(V) = T_{C^o}(R_j^i(V)) \).

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$U = \mathbb{P}^{n-1} - \mathbb{P}(D), \quad b_i(U) := \dim H^i(U, \mathbb{C})$.

In fact: $b_i(U) = b_i(V) - b_{i-1}(V) + \ldots + (-1)^ib_0(V)$.
Let \( c^{(j)}_i \) denote the coefficient of \( t^i \) in

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c^{(j)}(t) = \prod_{k=1}^{j+1} (1 - kt)^{-1} b_{j+1-k}(U).
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**Theorem (B.):** *Any Schur polynomial of weight* 

\[ < \text{codim } R^j_1(U) =: q_j \text{ in } c^{(j)}_1, \ldots, c^{(j)}_{q_j-1} \text{ is non-negative.} \]
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First few Schur polys: $c_1; c_2, c_1^2 - c_2; c_3, c_1c_2 - c_3, c_1^3 - 2c_1c_2 + c_3, \ldots$
**Proof:** (Method of Popa-Lazarsfeld for $h^{p,q}$ of irregular varieties.)
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$$0 \longrightarrow \mathcal{O}_\mathbf{P}(-n + 1) \otimes H^0(U) \overset{\phi_0}{\longrightarrow} \mathcal{O}_\mathbf{P}(-n + 2) \otimes H^1(U) \overset{\phi_1}{\longrightarrow} \ldots$$

$$\ldots \overset{\phi_{n-2}}{\longrightarrow} \mathcal{O}_\mathbf{P} \otimes H^{n-1}(U) \longrightarrow \mathcal{F} \longrightarrow 0.$$
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$\ldots \phi_{n-2} \to \mathcal{O}_\mathbf{P} \otimes H^{n-1}(U) \to \mathcal{F} \to 0.$

Recognize this as the linear locally free resolution of $\mathcal{F}$ constructed via BGG by Eisenbud-Popescu-Yuzvinsky.
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Chern poly of $\mathcal{F} = c^{(n-2)}(t)$.

Fulton-Lazarfeld: For a globally generated vector bundle, Chern classes and Schur polynomials in these are nonnegative. □
Corollary: $P(R^i_j(U))$ is the support of the ideal generated by the minors of size $\beta_i + 1 - j$ of $\phi_i$. 

(Surprisingly, unknown except for $R^i_j(U)$, Matei-Suciu.) Other implications for $R^i_j(U)$: codimension bounds, connectedness criterion, propagation. (Applications of results of Buchsbaum - Eisenbud, Fulton - Lazarsfeld, resp. Tchernev - Weyman.)
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III. Geometry

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(1) *Minimal log resolution* of \((\mathbb{C}^n, D)\) (De Concini - Procesi, Schechtman - Terao - Varchenko):

\[ \mu^* D = \sum_{Z = \text{dense edge}} a_Z E_Z, \quad a_Z = \text{mult} D(Z) \]

\[ \text{K}_{\mu} = \sum_{Z = \text{dense edge}} k_Z E_Z, \quad k_Z = \text{codim} Z - 1. \]
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$$K_\mu = \sum_{Z=\text{dense edge}} k_Z E_Z, \quad k_Z = \text{codim} Z - 1.$$
(2) Multiplier ideals: $\mathcal{I}(\mathbb{C}^n, cD)$

\[ \mathcal{I}(\mathbb{C}^n, cD) := \mu^* \mathcal{O}_Y (K_{\mu} - \lfloor c \mu^* D \rfloor) \]

Musta\c{t}a, Teitler:

$\mathcal{I}(\mathbb{C}^n, cD) = \bigcap_{Z = \text{dense edge}} I \lfloor c \cdot \text{mult} D (Z) \rfloor - \text{codim} (Z) + 1 Z$.

Jumping numbers:

$c \in \mathbb{R}^+$ such that $\mathcal{I}(\mathbb{C}^n, cD) \subsetneq \mathcal{I}(\mathbb{C}^n, (c-\epsilon)D) \forall \epsilon > 0$.

Theorem (B. - M. Saito):

Jumping numbers of hyperplane arrangements are given by explicit combinatorial formula.
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\mathcal{I}(\mathbb{C}^n, cD) = \bigcap_{Z=\text{dense edge}} I_Z^{\lfloor c \cdot \text{mult}_D(Z) \rfloor - \text{codim}(Z) + 1}.
\]

Jumping numbers: \( c \in \mathbb{R}_+ \) such that

\[
\mathcal{I}(\mathbb{C}^n, cD) \subsetneq \mathcal{I}(\mathbb{C}^n, (c - \epsilon)D) \quad \forall \epsilon > 0.
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**Theorem (B. - M. Saito):** *Jumping numbers of hyperplane arrangements are given by explicit combinatorial formula.*
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**Theorem (B. - M. Saito):** *The Hodge spectrum of a hyperplane arrangement, i.e. the numbers

$$n_c := \sum_{i \in \mathbb{Z}} (-1)^i \dim Gr^{[n-c]} F H^i(M_f,0, C)_{e^{2\pi i c}}, \quad c \in \mathbb{Q},$$

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admit explicit combinatorial formulas.
Proof. Use local systems, Hirzebruch-Riemann-Roch, to reduce to intersection numbers on the minimal log resolution. □
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(4) The topological zeta function (Denef - Loeser):

\[ Z_f^{top}(s) := \sum_{I \subset \mathcal{L}(D)^{dense}} \chi(E^\circ_I) \cdot \prod_{Z \in I} \frac{1}{azs + kZ + 1}, \]
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eigenvalue of monodromy.
(5) The *Bernstein - Sato polynomial* of $f$, also called the *$b$-function*:

$$b_f(s) = \text{non-zero monic polynomial}$$

is the non-zero monic polynomial $b_f \in \mathbb{C}[s]$ of minimal degree among those for which

$$b_f(s)f_s = P \cdot f_s + 1$$

for some $P \in \mathbb{C}[x_1, \ldots, x_n, \partial \partial x_1, \ldots, \partial \partial x_n, s]$.

Example (Walther, 2003):

$$f = \text{central generic hyperplane arrangement of degree } d \Rightarrow b_f(s) = (s+1)^{n-1}2^{d-2}\prod_{j=1}^{n}(s+jd).$$
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b_f(s) = (s + 1)^{n-1} \prod_{j=n}^{2d-2} \left( s + \frac{j}{d} \right).
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n/d - Conjecture (B. - Mustaţă - Teitler):

If $f$ is a central essential indecomposable hyperplane arrangement in $\mathbb{C}^n$ of degree $d$, then $b_f(\frac{-n}{d}) = 0$.

Even this one root of $b_f$ is very useful:

Proposition (BMT): If n/d-Conjecture is true, it implies the Strong Monodromy Conjecture for all hyperplane arrangements:

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Theorem (B.-Saito-Yuzvinsky):

- reduced $D$, $n \leq 3$;
- reduced $D$, $(n, d) = 1$, and one hyperplane is in general position.

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The Strong Monodromy Conjecture holds for:
- reduced hyperplane arrangements in $\leq 3$ variables;
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Proof:
Proof: Griffiths and Deligne related pole order filtration with Hodge filtration on $H^*(U, \mathbb{C})$ when $U =$ complement of a snc divisor $\mathbb{P}(D)$ in $\mathbb{P}^n$. 

Brieskorn lattices produce roots of $b_f$ (Malgrange, Saito). A form $df_1f_1 \wedge ... \wedge df_nf_n$ will have the right pole order to produce the root $-n/d$ if the hyperplanes $f_1,...,f_n$ satisfy a combinatorial condition. □
Proof: Griffiths and Deligne related pole order filtration with Hodge filtration on $H^*(U, \mathbb{C})$ when $U =$ complement of a snc divisor $\mathbb{P}(D)$ in $\mathbb{P}^n$. Dimca and M. Saito extended this to arbitrarily singular $D$ and to non-trivial local systems $\mathcal{V}$ by relating pole order filtration with Brieskorn lattices of Milnor fiber.
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(5) *Questions*: What about subspace arrangements?
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Cohomology ring $H^*(V, \mathbb{Z})$ of complement is still combinatorial:


Canonical log resolutions, log canonical thresholds, Monodromy Conjecture, ... ?
IV. Arithmetic

(1) The $p$-adic zeta function.

(Igusa)

$f \in \mathbb{Z}[[x_1, \ldots, x_n]]$;

$p$ = prime number;

$N_m(a) := \{ x \in (\mathbb{Z}/p^m\mathbb{Z})^n | f(x) \equiv a \text{ modulo } p^m \}$.

$P_f(t) := 1 + N_1(0) p^n t + N_2(0) p^{2n} t^2 + N_3(0) p^{3n} t^3 + \ldots$.

$P_f(t)$ is a rational function (Igusa).

$Z_p f(s) := p^{s - (p^s - 1)} P_f(p^{-s}) = \int_{\mathbb{Z}_p^n} |f(x)|^s dx$.
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Nero Budur (University of Notre Dame) Hyperplane Arrangements
Poles of $Z^p_f(s)$ give asymptotics of $N_m(a)$.

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All proofs for $Z^{top}_f(s)$ seems to translate well for $Z^p_f(s)$. Hence, we also obtain the $p$-adic SMC for the hyperplane arrangement cases covered by BSY.
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- monomial ideals (Howald - Mustaţă - Yuen).
(2) Question:
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~ THANK YOU! ~