ALGEBRAIC GEOMETRY

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Contents

1. Introduction

These notes are for a first graduate course on algebraic geometry. It is assumed that the students are not familiar with algebraic geometry so we have started from scratch. I have taken a moderate approach emphasising both geometrical and algebraic thinking. We have borrowed few main theorems of commutative algebra but rigorous proofs are given for the rest, except the sections on sheaves and cohomology. I believe this is the best way of introducing algebraic geometry rather than starting with schemes. One has to accept the fact that it is not possible to teach too much advanced algebraic geometry in two months. Students who will continue with algebraic geometry or number theory need to have another course on schemes. A student who understands these lectures should have no problem learning schemes.

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Warning to students: the notes don't exactly correspond to the lectures given though the difference is quite small.

I would like to thank Burt Totaro and those students whose comments and questions helped to improve these notes. Any more comment/correction is welcome.

2. Affine varieties

Throughout this document, we assume k to be an algebraically closed field unless stated otherwise. We denote $k[t_1, \dots, t_n]$ for the ring of polynomials over k with the variables $t_1, \dots, t_n$. It is well-known that this a unique factorisation domain (UFD).

We assume good knowledge of commutative algebra, that is, general properties of rings, ideals, modules, fields, etc. Corresponding to the algebraic object $k[t_1, \cdots, t_n]$ we have the following geometric object.

Definition 2.1 (Affine space). *The n-dimensional affine space $\mathbb{A}_k^n$ is defined as*

$$\mathbb{A}_k^n = \{x = (x_1, \dots, x_n) \mid x_i \in k\}$$

We call $\mathbb{A}_k^1$ the affine line and $\mathbb{A}_k^2$ the affine plane. For the moment the word n-dimensional is just a terminology but later on we will see that $\mathbb{A}_k^n$ really has dimension n.

We are interested in some special subsets of the n-dimensional affine space $\mathbb{A}_k^n$.

Definition 2.2. *An affine algebraic set X is a set*

$$X = V(f_1, \dots, f_m) = \{x \in \mathbb{A}_k^n \mid f_i(x) = 0, \forall i\}$$

where $f_i \in k[t_1, \dots, t_n]$. That is, an affine algebraic set is the set of common solutions of finitely many polynomials.

Polynomials are among the most simple kind of functions, so it is not a surprise that algebraic geometry is fundamentally related to number theory and all other kinds of geometry, physics, etc. Though polynomials are simple but the study of algebraic sets is among the hardest and most fascinating subjects of mathematics.

Example 2.3. *In this example let $k = \mathbb{Q}$ and let $f = t_1^d + t_2^d - 1$. What is $X = V(f) \subseteq \mathbb{A}_{\mathbb{Q}}^2$? This is Fermat's last theorem.*

One can also define algebraic sets using ideals rather than polynomials.

Definition 2.4. *For an ideal I of $k[t_1, \dots, t_n]$ define its affine algebraic set as*

$$V(I) = \{x \in \mathbb{A}_k^n \mid f(x) = 0, \forall f \in I\}$$

The algebraic set of an ideal is not a new object by the following theorem.

Theorem 2.5 (Hilbert basis theorem)**.** *Every ideal I of $k[t_1,\dots,t_n]$ is finitely generated, that is, there are $f_1,\dots,f_m \in I$ such that $I = \langle f_1,\dots,f_m\rangle$.*

In other words, the ring $k[t_1,\dots,t_n]$ is noetherian.

Example 2.6. *If $I = \langle f_1,\dots,f_m\rangle$, then prove that $V(I) = V(f_1,\dots,f_m)$.*

Example 2.7. *Affine algebraic subsets of $\mathbb{A}^1_k$ are $\emptyset$, $\mathbb{A}^1_k$ and its finite subsets.*

Example 2.8. *A line in $\mathbb{A}^2_k$ is given by a nonzero linear polynomial. It is easy to see that $V(t_1t_2)$ is the union of two lines, the two axis, and $V(t_1,t_2)$ is the origin $(0,0)$.*

The reason we use ideals rather than polynomials is that it is more convenient to relate algebraic sets and ideals.

Theorem 2.9. *Let I, J be ideals of $k[t_1,\dots,t_n]$. Then,*
$V(IJ) = V(I\cap J) = V(I)\cup V(J)$ and $V(I+J) = V(I)\cap V(J)$.

Proof. Straightforward. □

When we are given an affine algebraic set we can try to recover its ideal. It is not possible to recover the ideal but we can recover the biggest possible ideal which defines the algebraic set.

Definition 2.10. *Let $X \subseteq \mathbb{A}^n_k$ be an affine algebraic set. Define its ideal I_X as*

$$I_X = \{f \in k[t_1,\dots,t_n] \mid f(x) = 0, \forall x \in X\}$$

Exercise 2.11. *If $X = V(I)$, prove that $I \subseteq I_X$.*

Example 2.12. *Let $X = V(I) \subset \mathbb{A}^1_k$ where $I = \langle t^2\rangle$. Then, $I_X = \langle t\rangle$. In particular, this shows that some times $I \neq I_X$.*

Exercise 2.13. *Let $X', X'' \subseteq \mathbb{A}^n_k$ be affine algebraic sets. Prove that if $X' \subseteq X''$ then $I_{X''} \subseteq I_{X'}$. Prove that $I_{X'\cup X''} = I_{X'} \cap I_{X''}$.*

An open subset of $\mathbb{A}^n_k$ is defined to be the complement of an affine algebraic set , that is, $U = \mathbb{A}^n_k - X$ where X is an affine algebraic set. So, we are considering the affine algebraic sets as closed subsets. this makes $\mathbb{A}^n_k$ into a topological space which is called the Zariski topology. This induces a topology on every subset of the affine space and we always refer to it as the Zariski topology.

A topological space T is called irreducible if $T = T_1 \cup T_2$ is a union of closed subsets then $T = T_1$ or $T = T_2$.

Definition 2.14. *An irreducible affine algebraic set (with respect to the Zariski topology) is called an affine algebraic variety.*

Theorem 2.15. *Let X be an affine algebraic set. Then, X is an affine algebraic variety iff I_X is a prime ideal.*

Proof. Suppose that $X \subseteq \mathbb{A}_k^n$ is an affine algebraic variety. Assume that $fg \in I_X$ for $f, g \in k[t_1, \dots, t_n]$. Then,

$$X = \{x \in X \mid f(x) = 0\} \cup \{x \in X \mid g(x) = 0\}$$

where the two latter sets are closed subsets of X. So, X should be equal to one of them and so $f \in I_X$ or $g \in I_X$.

The converse is similarly proved. □

The ring $k[t_1, \dots, t_n]$ considered as an ideal of itself is not defined to be prime, so in the theorem or other places we can exclude the case $X = \emptyset$.

To see the relation between affine algebraic sets and ideals more clearly we need to borrow the following important theorem.

Theorem 2.16 (Hilbert Nullstellensatz)**.** *Let I be an ideal of $k[t_1, \dots, t_n]$. Then,*
(i) if $X = V(I)$, then $I_X = \sqrt{I}$,
(ii) any maximal ideal of $k[t_1, \dots, t_n]$ is of the form $\langle t_1 - x_1, \dots, t_n - x_n \rangle$,
(iii) $V(I) = \emptyset$ iff $I = k[t_1, \cdots, t_n]$.

See [1] for a proof.

Theorem 2.17. *Every affine algebraic set is a union of finitely many affine algebraic varieties (decomposition into irreducible components).*

Proof. If an affine algebraic set is not irreducible then $X = X_1 \cup X_2$ where X_1, X_2 are closed subsets of X. If X_2 is not irreducible, then $X = X_1 \cup X_3 \cup X_4$ and so on. This stops or we find a decreasing sequence $\cdots X'' \subsetneq X' \subsetneq X$ of closed subsets. Therefore, by the Nullstellensatz we find an increasing sequence of ideals $I_X \subsetneq I_{X'} \subsetneq \cdots$ which contradicts the fact that $k[t_1, \dots, t_n]$ is a noetherian ring. □

Theorem 2.18. *An affine algebraic set X is a single point iff I_X is a maximal ideal.*

Proof. Use the Nullstellensatz theorem. □

So, we have a one-to-one correspondence between the prime ideals of $k[t_1, \dots, t_n]$ and the affine algebraic varieties $X \subseteq \mathbb{A}_k^n$. We also have a one-to-one correspondence between the maximal ideals of $k[t_1, \dots, t_n]$ and the points of $\mathbb{A}_k^n$.

Example 2.19. *An irreducible polynomial $f \in k[t_1, \cdots, t_n]$ defines a prime ideal $I = \langle f \rangle$ and so defines an affine algebraic variety $X = V(f) \subseteq \mathbb{A}_k^n$. If $n = 2$, this is called a curve, if $n = 3$ it is called a surface, and if $n \geq 4$ it is called a hypersurface.*

Example 2.20. *Any $0 \neq f \in k[t_1, \cdots, t_n]$ can be decomposed into irreducible factors $f = f_1 \cdots f_m$. Therefore, $V(f) = V(f_1) \cup \cdots \cup V(f_m)$ is the decomposition into irreducible components.*

Example 2.21. *Form commutative algebra, we know that a prime ideal in $k[t_1, t_2]$ is either 0, $\langle f \rangle$ for an irreducible f, or a maximal ideal $\langle t_1 - x_1, t_2 - x_2 \rangle$. Therefore, an affine algebraic variety in $\mathbb{A}_k^2$ is either $\emptyset$, $\mathbb{A}_k^2$, a curve or a point.*

Exercise 2.22. *Prove the previous example directly.*

The set-theoretic product $\mathbb{A}_k^n \times \mathbb{A}_k^m$ may be identified with $\mathbb{A}_k^{n+m}$ by identifying $(x_1, \ldots, x_n), (y_1, \ldots, y_m)$ with $(x_1, \ldots, x_n, y_1, \ldots, y_m)$. Now if $X \subseteq \mathbb{A}_k^n$ and $Y \subseteq \mathbb{A}_k^m$ are affine algebraic sets, then $X \times Y \subseteq \mathbb{A}_k^{n+m}$ is an affine algebraic set in the obvious way.

Definition 2.23. *For an affine algebraic set $X \subseteq \mathbb{A}_k^n$, its coordinate ring is defined as $k[X] = k[t_1, \ldots, t_n]/I_X$. This ring is a finitely generated k-algebra. Elements of this ring are called regular functions on X, they can be viewed as functions $X \to k$ and each one can be represented by a polynomial but not in a unique way.*

Theorem 2.24. *An affine algebraic set $X \subseteq \mathbb{A}_k^n$ is an affine algebraic variety iff $k[X]$ is an integral domain. Moreover, X is a point iff $k[X] = k$.*

Proof. Clear. □

Definition 2.25. *Let $X \subseteq \mathbb{A}_k^n$ and $Y \subseteq \mathbb{A}_k^m$ be affine algebraic sets. A regular map $\phi\colon X \to Y$ is a map given by $\phi = (f_1, \ldots, f_m)$ where f_i are regular functions on X. ϕ is an isomorphism if it has an inverse which is also a regular map.*

Example 2.26. *Regular functions on an affine algebraic set X are regular maps from X to $k = \mathbb{A}_k^1$.*

Projections $\phi\colon \mathbb{A}_k^n \to \mathbb{A}_k^m$ defined by $\phi = (t_1, \ldots, t_m)$, where $m \leq n$, are regular maps. If $X \subseteq \mathbb{A}_k^n$ is an affine algebraic set, the restriction map $\phi|_X\colon X \to \mathbb{A}_k^m$ is also a regular map.

Every regular map $\phi\colon X \to Y$ of affine algebraic sets gives rise to a homomorphism of k-algebras $\phi^*\colon k[Y] \to k[X]$ by combining regular functions on Y with ϕ.

For an affine algebraic set X and an ideal I of $k[X]$ we define

$$V_X(I) = \{x \in X \mid f(x) = 0, \forall f \in I\}$$

Theorem 2.27. *Let $\phi\colon X \to Y$ be a regular map of affine algebraic sets. Then,*
(i) ϕ is a continuous map of topological spaces,
(ii) ϕ^* is injective iff $\phi(X)$ is dense in Y,
(iii) ϕ is an isomorphism iff $\phi^*\colon k[Y] \to k[X]$ is an isomorphism of k-algebras.

Proof. (i) This is the case because for an ideal I of $k[Y]$, we have $\phi^{-1}V_Y(I) = V_X(Ik[X])$. In other words, inverse of closed subsets are closed and so ϕ is continuous.

(ii) Suppose that $\phi(X)$ is dense in Y and $\phi^*(f) = 0$. Then, $\phi(X) \subseteq V_Y(f)$ which is possible only if $f = 0$. Conversely, suppose that ϕ^* is injective and assume that $\phi(X) \subseteq V_Y(I)$ for some ideal I of $k[Y]$. Obviously, $I \subseteq \ker \phi^* = 0$, so $\phi(X)$ must be dense in Y.

(iii) Exercise. □

The theorem suggests that affine algebraic sets are uniquely determined by their coordinate rings. So, one should be able to discover all the properties of an affine algebraic set from its coordinate ring. This led to the revolution by Grothendieck which completely transformed algebraic geometry and related subjects. The idea is that why not look at any commutative ring and define some space for it and study its geometry. Such spaces are called schemes.

Example 2.28. *Let $X = V(t_1t_2 - 1) \subseteq \mathbb{A}^2_k$. The map to $\mathbb{A}^1_k$ given by $\phi = t_1$ is regular and injective but not surjective so not an isomorphism. Describe the ring homomorphism.*

Example 2.29. *The regular map $\phi\colon \mathbb{A}^1_k \to Y$ given by $\phi = (t^2, t^3)$ where $Y = V(s_2^2 - s_1^3) \subseteq \mathbb{A}^2_k$ (cusp singularity) is a 1-1 regular map however, it is not an isomorphism. In fact, the corresponding homomorphism $\phi^*\colon k[Y] = k[s_1, s_2]/\langle s_2^2 - s_1^3\rangle \to k[t]$ is determined by $\phi^*(s_1) = t^2$ and $\phi^*(s_2) = t^3$. So t cannot be in the image of ϕ^* hence it is not an isomorphism. If one tries to get an inverse it should be as $\psi = s_2/s_1$ which is not a regular function. Later we see that ψ is a rational function.*

Let $X \subseteq \mathbb{A}^n_k$ be an affine algebraic variety. Since $k[X]$ is an integral domain, its field of fractions $k(X)$ can be constructed which is called the field of rational functions of X or the function field of X. Each element of $k(X)$ can be represented by f/g such that $g \neq 0$ and $f, g \in k[X]$. Equivalently, we can write it as f/g where $f, g \in k[t_1, \dots, t_n]$ such that

$g \notin I_X$. It can be considered as a "function" $\pi\colon X \dashrightarrow k$ which may not be defined everywhere. We say that $\pi = f/g$ is defined or regular at $x \in X$ if we can find h, e such that $\pi = f/g = h/e$ and $e(x) \neq 0$.

Example 2.30. *(i) Regular functions are also rational functions. (ii) When $X = \mathbb{A}^n_k$, any element of the field $k(X) = k(t_1, \ldots, t_n)$ is a rational function on X. (iii) The function ψ defined in Example 2.29 is a rational function on $Y = V(s_2^2 - s_1^3) \subseteq \mathbb{A}^2_k$.*

Theorem 2.31. *Let X be an affine algebraic variety. A rational function $\pi\colon X \dashrightarrow k$ is regular everywhere iff it is a regular function.*

Proof. Suppose that π is a rational function which is regular everywhere. Then, for each $x \in X$ we can write $\pi = f_x/g_x$ such that $g_x(x) \neq 0$. Now if I is the ideal in $k[X]$ generated by all g_x, then $V_X(I) = \emptyset$ and so $I = k[X]$ by Hilbert Nullstellensatz theorem. Thus, $\sum_{i=1}^m h_i g_{x_i} = 1$ for some $h_i \in k[X]$ and finitely many points $x_1, \ldots, x_m \in X$. Now $\pi = \pi \sum h_i g_{x_i} = \sum \pi h_i g_{x_i} = \sum h_i f_{x_i}$ where $\pi = f_{x_i}/g_{x_i}$. □

Let X be an affine algebraic variety and $\pi \in k(X)$ a rational function. The domain of π is the set of points of X at which π is regular.

Theorem 2.32. *Let X be an affine algebraic variety. Then, the domain of a rational function $\pi\colon X \dashrightarrow k$ is a nonempty open subset of X.*

Proof. Since we can write $\pi = f/g$ such that $g \neq 0$, the domain U is not empty. Now if I is the ideal in $k[X]$ generated by all g of different way of writing $\pi = f/g$, then $V_X(I) = X - U$. □

Exercise 2.33. *Let X be an affine algebraic variety. Prove that*
(i) the intersection of finitely many nonempty open subsets of X is open and nonempty,
(ii) if U is a nonempty open subset of X, then it is dense in X,
(iii) the domain of finitely many rational functions is also open and nonempty.

Exercise 2.34. *Prove that if two rational functions of an affine algebraic variety are equal on a nonempty open subset, they should be equal.*

Definition 2.35. *Let $X \subseteq \mathbb{A}^n_k$ be an affine algebraic variety and $Y \subseteq \mathbb{A}^m_k$ an affine algebraic set. A rational map $\pi\colon X \dashrightarrow Y$ is given by $\pi = (\pi_1, \ldots, \pi_m)$ where $\pi_i \in k(X)$ and such that $\pi(x) = (\pi_1(x), \ldots, \pi_m(x)) \in Y$ if all the π_i are defined at x, and we say that π is defined or regular at x. In particular, $\pi(X) = \{\pi(x) \mid \pi \text{ regular at } x\}$.*

Definition 2.36. *A rational map $\pi\colon X \dashrightarrow Y$ such that $\pi(X)$ is dense in Y gives a homomorphism of fields $\pi^*\colon k(Y) \to k(X)$. π is called a birational isomorphism if Y is also a variety and such that there is π^{-1} which is the inverse of π where they are defined. We also call X and Y birational.*

Exercise 2.37. *Prove that a rational map $\pi\colon X \dashrightarrow Y$ is a birational isomorphism iff π^* is an isomorphism of fields.*

Example 2.38. *Let $\pi\colon \mathbb{A}^1_k \to Y$ be the regular map given by $\pi = (t^2, t^3)$ where $Y = V(s_2^2 - s_1^3) \subseteq \mathbb{A}^2_k$. We proved that π is not an isomorphism. Now $\theta = s_2/s_1$ is the inverse of π which is not regular but rational. So, π is a birational isomorphism.*

Example 2.39. *Let $Y = V(f) \subset \mathbb{A}^2_k$ be a variety defined by f of degree 2 and after a linear change of variables we can assume that $(0,0) \in Y$. Let L_s be the line defined by $V(t_2 - st_1)$ where $s \in k$. This line intersects Y at $(0,0)$. Since f has degree 2, it intersects Y at another point y_s except for finitely many values of s. To see this one needs to solve the equation $g(t_1) = f(t_1, st_1) = 0$. Except for finitely many $s \in k$, $g(t_1)$ is of degree 2 and 0 is one of its solutions, so $g(t_1) = t_1 h(t_1)$ for some polynomial $h(t_1) = A(s)t_1 + B(s)$ which is linear in t_1. Thus, the other root of g is given by a rational function $\pi_1 = -B(s)/A(s)$. The other coordinate of y_s is given by the rational function $\pi_2 = s\pi_1$. In short, we have defined a rational map $\pi\colon \mathbb{A}^1_k \dashrightarrow Y$ given by $\pi = (\pi_1, \pi_2)$. Now define $\theta\colon Y \dashrightarrow \mathbb{A}^1_k$ by $\theta = t_2/t_1$. We see that π and θ are inverse of each other.*

Exercise 2.40. *Let $Y = V(t_1^2 + (t_2 - 1)^2 - 1) \subset \mathbb{A}^2_k$. Find the rational maps π and θ of the previous example explicitly.*

We can also define regular and rational maps on open subsets of affine algebraic sets.

Definition 2.41. *A quasi-affine algebraic set $X \subseteq \mathbb{A}^n_k$ is an open subset of an affine algebraic set. A regular function on X is a function $\phi\colon X \to k$ such that for every $x \in X$, there is a neighborhood U of x, and $f, g \in k[t_1, \dots, t_n]$ such that on U, ϕ and f/g are equal, in particular, g has no zero on U. The set of regular functions on X is denoted by $k[X]$ which is a k-algebra.*

If X is irreducible, a rational function $\pi\colon X \dashrightarrow k$ on X is the equivalence class of a regular function on some open subset of X in the sense that if ϕ_U and ϕ_V are regular functions on the open subsets U and V respectively, then ϕ_U is equivalent to ϕ_V if $\phi_U|_{U\cap V} = \phi_V|_{U\cap V}$. The set of rational functions on X is denoted by $k(X)$ which is a field and is called the function field of X.

Exercise 2.42. *If* $X \subseteq \mathbb{A}^n_k$ *is an affine algebraic set, prove that regular functions on* X *in the previous definition are the same as before.*

3. Quasi-projective varieties.

A (commutative) graded ring is a commutative ring $S = \bigoplus_{d=0}^{+\infty} S_d$ such that S_d are abelian groups and $S_d S_{d'} \subseteq S_{d+d'}$. Elements of S_d are called homogeneous of degree d. An ideal I of S is called homogeneous if $I = \bigoplus_{d=0}^{+\infty}(S_d \cap I)$. This is equivalent to I being generated by homogeneous elements. In particular, $S_+ = \bigoplus_{d=1}^{+\infty} S_d$ is a homogeneous ideal.

Exercise 3.1. *Prove that intersection, sum, product and radical of homogeneous ideals are again homogeneous.*

Example 3.2. *The most important example for us is the polynomial ring $S = k[s_0, \dots, s_n]$ which has a natural graded structure. By Hilbert basis theorem, every homogeneous ideal of this ring is finitely generated.*

We say that $(x_0, \dots, x_n), (y_0, \dots, y_n) \in \mathbb{A}_k^{n+1}$ are equivalent if there is a nonzero $a \in k$ such that $(y_0, \dots, y_n) = (ax_0, \dots, ax_n)$. The equivalence class of $(x_0, \dots, x_n)$ is denoted by $(x_0 : \dots : x_n)$.

Definition 3.3 (Projective space)**.** *The n-dimensional projective space is defined as*

$$\mathbb{P}_k^n = \{x = (x_0 : \dots : x_n) \mid x_i \in k \textit{ and some } x_i \neq 0\}$$

We call $\mathbb{P}_k^1$ the projective line and call $\mathbb{P}_k^2$ the projective plane.

The projective line $\mathbb{P}_k^1 = \mathbb{A}_k^1 \cup \{(0:1)\}$ which is like saying that $\mathbb{P}_k^1$ is obtained by adding the point at infinity to $\mathbb{A}_k^1$.

There are good reasons to introduce the projective space. It is in some sense the perfect world where objects present whatever they have got. For example later on we will see that any two projective curves in $\mathbb{P}_k^2$ intersect. This is clearly not true in $\mathbb{A}_k^2$. One can think of the projective space as a compactified version of the affine space.

Now let $F \in k[s_0, \dots, s_n]$ and $x \in \mathbb{P}_k^n$. Then, $F(x) = 0$ is well-defined iff F is homogeneous. Even if F is homogeneous, F does not define a function $\mathbb{P}_k^n \to k$.

Definition 3.4. *Let I be a homogeneous ideal of $k[s_0, \dots, s_n]$. Define*

$$V(I) = \{x \in \mathbb{P}_k^n \mid \forall F \in I \textit{ homogeneous}, F(x) = 0\}$$

If $F_1, \dots, F_m$ are homogeneous polynomials generating I, then $V(I) = V(F_1, \dots, F_m)$. A projective algebraic set X in $\mathbb{P}_k^n$ is defined to be $X = V(I)$ for some homogeneous ideal I of $k[s_0, \dots, s_n]$.

Example 3.5. *Let $M_{2\times 2}$ be the set of 2×2 matrices over k. Any nonzero element of $M_{2\times 2}$ determines a point of $\mathbb{P}^3_k$. Now all such matrices which are not invertible define a projective algebraic set in $\mathbb{P}^3_k$.*

Exercise 3.6. *(i) For homogeneous ideals of $k[s_0,\dots,s_n]$, prove that $V(IJ)=V(I\cap J)=V(I)\cup V(J)$ and $V(\sum I_l)=\bigcap V(I_l)$. (ii) Prove that $\mathbb{P}^n_k$ is a topological space by taking its closed subsets to be projective algebraic sets. This induces a topology on each subset of $\mathbb{P}^n_k$ which we call the Zariski topology.*

A quasi-projective algebraic set is an open subset of a projective algebraic set. A projective variety is an irreducible projective algebraic set. A quasi-projective variety is an open subset of a projective variety. For a projective algebraic set $X\subseteq\mathbb{P}^n_k$, define its ideal I_X as the ideal generated by

$$\{F\in k[s_0,\dots,s_n]\mid F \text{ is homogeneous, } F(x)=0 \text{ for all } x\in X\}$$

I_X is a homogeneous ideal. Now define the coordinate ring of X as $S[X]=k[s_0,\dots,s_n]/I_X$ which is a graded ring.

Exercise 3.7. *Let $X',X''\subseteq\mathbb{P}^n_k$ be projective algebraic sets. Prove that if $X'\subseteq X''$ then $I_{X''}\subseteq I_{X'}$. Prove that $I_{X'\cup X''}=I_{X'}\cap I_{X''}$.*

Theorem 3.8. *Let I be a homogeneous ideal of $k[s_0,\dots,s_n]$. Then,*
(i) $V(I)=\emptyset$ iff $S_+\subseteq\sqrt{I}$,
(ii) If $X=V(I)\neq\emptyset$, then $I_X=\sqrt{I}$.

Proof. (i) If $V(I)=\emptyset$, then the algebraic set defined by I in $\mathbb{A}^{n+1}_k$, that is, $Y=V(I)\subseteq\mathbb{A}^{n+1}_k$ is empty or it is the origin. In any case, this implies that $S_+\subseteq\sqrt{I}$.

The inverse: if $(x_0:\dots:x_n)\in V(I)$, then every homogeneous element of I vanishes at $(x_0,\dots,x_n)$. Since I is homogeneous, so every element of I vanishes at $(x_0,\dots,x_n)$. So, $(0,\dots,0)\neq(x_0,\dots,x_n)\in Y$. This is not possible because $S_+\subseteq\sqrt{I}$.

(ii) Let again $Y=V(I)\subseteq\mathbb{A}^{n+1}_k$. Note that the map $\xi\colon Y-\{0\}\to X$ given by $\xi(x_0,\dots,x_n)=(x_0:\dots:x_n)$ is surjective. On the other hand, for a homogeneous polynomial F, $F(x_0:\dots:x_n)=0$ iff $F(x_0,\dots,x_n)=0$. So, $F\in I_X$ iff $F\in I_Y$. Since I_X and $I_Y=\sqrt{I}$ are homogeneous, $I_X=I_Y=\sqrt{I}$. □

Affine and projective algebraic sets are closely related. Projective algebraic sets have a more complicated global structure and we usually try to understand this structure by covering a projective algebraic set by affine algebraic sets. This is the main reason to define differential forms and cohomology groups later on.

Theorem 3.9. *Every projective algebraic set is covered by finitely many affine algebraic sets (for the moment this is just set-theoretic).*

Proof. If $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n_k$ such that $x_i \neq 0$, then we can uniquely write $x = (x'_0 : \cdots : x'_{i-1} : 1 : x'_{i+1} : \cdots : x'_n)$ where $x'_j = x_j/x_i$. Let $U_i = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n_k \mid x_i \neq 0\}$. This can be identified with $\mathbb{A}^n_k$. So, $\mathbb{P}^n_k$ is covered by $n+1$ copies of $\mathbb{A}^n_k$. On the other hand, an algebraic set $X \subseteq \mathbb{P}^n_k$ is covered by $X \cap U_i$. Suppose that $X = V(F_1, \dots, F_m)$. Then, $X \cap U_i = V(f_1, \dots, f_m) \subseteq \mathbb{A}^n_k$ where $f_j = F_j(s_0, \dots, s_{i-1}, 1, s_{i+1}, \dots, s_n)$. Here $s_0, \dots, s_{i-1}, s_{i+1}, \dots, s_n$ are taken as the variables on $\mathbb{A}^n_k$. □

For $f \in k[t_1, \dots, t_n]$ of degree d, define the homogeneous polynomial F associated to f by $s_0^d f(\frac{s_1}{s_0}, \dots, \frac{s_n}{s_0})$ which is in $k[s_0, \dots, s_n]$. Conversely, for a homogeneous $G \in k[s_0, \dots, s_n]$, we can associated $g = G(1, t_1, \dots, t_n) \in k[t_1, \dots, t_n]$.

Example 3.10. *Projectivisation of an affine algebraic set: let $X = V(I) \subseteq \mathbb{A}^n_k$ be an affine algebraic set. By identifying $\mathbb{A}^n_k$ with U_0 as in the proof of Theorem 3.9 we can take the closure $\overline{X}$ in $\mathbb{P}^n_k$ which is by definition the intersection of all projective algebraic sets in $\mathbb{P}^n_k$ containing X. Let J be the homogeneous ideal of $k[s_0, \dots, s_n]$ generated by all the F associated to $f \in I$. We prove that $\overline{X} = V(J) \subseteq \mathbb{P}^n_k$. Let $x \in X$ and $G \in J$ homogeneous. Then, $G = \sum P_i F_i$ where F_i is associated to $f_i \in I$, and so $G(x) = 0$ because $F_i(x) = 0$ since $f_i(x) = 0$. In other words, $X \subseteq \overline{X} \subseteq V(J)$.*

If $\overline{X} \neq V(J)$, then $\overline{X} \subseteq V(\langle G \rangle + J) \subsetneq V(J)$ for some homogeneous polynomial G. Then $X = V(\langle g \rangle + I)$ where g is associated to G. Thus, $g \in \sqrt{I}$ and so $G^l \in J$ for some $l \in \mathbb{N}$. Therefore, $V(\langle G \rangle + J) = V(J)$, a contradiction.

As mentioned earlier, a homogeneous polynomial $F \in k[s_0, \dots, s_n]$ does not necessarily define a function on $\mathbb{P}^n_k$ or a subset of it. However, if we take two homogeneous polynomials $F, G \in k[s_0, \dots, s_n]$ of the same degree, by putting

$$(F/G)(x_0 : \cdots : x_n) = F(x_0, \dots, x_n)/G(x_0, \dots, x_n)$$

if $G(x_0, \dots, x_n) \neq 0$, we get a well-defined function $F/G \colon \mathbb{P}^n_k - V(G) \to k$.

Definition 3.11. *Let X be a quasi-projective algebraic set. A function $\phi \colon X \to k$ is called a regular function if for every $x \in X$, there is a neighborhood U of x, and homogeneous polynomials F, G of the same degree such that on U, ϕ and F/G are equal, in particular, G has no* *zero on U. The set of regular functions on X is denoted by $k[X]$ which is a k-algebra.*

If X is irreducible, a rational function $\pi\colon X \dashrightarrow k$ on X is the equivalence class of a regular function on some open subset of X in the sense that if ϕ_U and ϕ_V are regular functions on the open subsets U and V respectively, then ϕ_U is equivalent to ϕ_V if $\phi_U|_{U\cap V} = \phi_V|_{U\cap V}$. A rational function then is uniquely determined by some F/G where G is not identically zero on X. The set of rational functions on X is denoted by $k(X)$ which is a field and is called the function field of X.

Note that for a quasi-projective algebraic variety X, unlike the affine case, $k(X)$ is not necessarily the fraction field of $k[X]$.

Exercise 3.12. *Let $U \neq \emptyset$ be an open subset of a quasi-projective variety X. Prove that $k(U) = k(X)$.*

Example 3.13. *Let $X = \mathbb{P}^n_k$. We see that $k(X) \simeq k(t_1, \dots, t_n)$ where $t_i = s_i/s_0$.*

Example 3.14. *Let $X = \mathbb{P}^n_k$, we prove that $k[X] = k$. Let $\phi\colon X \to k$ be a regular function. Then, for $x \in X$ there is a neighborhood $x \in U$ and F, G homogeneous of the same degree in $k[s_0, \dots, s_n]$ such that $\phi = F/G$ on U, and G has no zero on U. We can assume that F, G have no common factor. Suppose that G is not a constant and let $x' \in V(G) - V(F)$. Then, there is a neighborhood $x' \in U'$ and F', G' homogeneous of the same degree in $k[s_0, \dots, s_n]$ such that $\phi = F'/G'$ on U', and G' has no zero on U', and F', G' have no common factor. So, F/G and F'/G' give the same values on $U \cap U' \neq \emptyset$. Since X is irreducible, this is possible only if $FG' - F'G = 0$. This is a contradiction because $F(x')G'(x') \neq 0$ but $F'(x')G(x') = 0$. So, F, G are constant and so is ϕ.*

Definition 3.15. *Let X and Y be quasi-affine or quasi-projective algebraic sets. A regular map $\phi\colon X \to Y$ is a continuous map (of Zariski topologies) such that it sends a regular function on a open subset $V \subseteq Y$ to a regular function on $\phi^{-1}V$, that is, if $f \in k[V]$ then the composition $f\phi \in k[\phi^{-1}V]$. The regular map $\phi\colon X \to Y$ is called an isomorphism if it has an inverse which is also a regular map.*

When we say a quasi-affine or a quasi-projective algebraic set is affine, we mean that it is isomorphic to an affine algebraic set in some affine space.

When X is irreducible, a rational map $\pi\colon X \dashrightarrow Y$ is the equivalence class of a regular map $\phi_U\colon U \to Y$ for some open subset $U \subseteq X$. We say that π is a birational isomorphism if it has a rational inverse θ. In this case, we also say that X and Y are birational.

Earlier we defined regular functions on an affine algebraic set. In Example 3.10, we also noted that an affine algebraic set can be embedded into a projective space set-theoretically.

Theorem 3.16. *Let $X \subseteq \mathbb{A}^n_k$ be a quasi-affine algebraic set and X' the corresponding quasi-projective algebraic set in $\mathbb{P}^n_k$ via the bijection $\phi\colon \mathbb{A}^n_k \to U_0 \subset \mathbb{P}^n_k$. Then, $\phi|_X\colon X \to X'$ is an isomorphism. So, quasi-affine algebraic sets are also quasi-projective.*

Proof. Clearly, $\phi|_X$ is a homeomorphism of topological spaces. Let $V \subseteq X$ be an open subset and $V' = \phi|_X(V)$. Then, a regular function on V is a function $V \to k$ which is locally given by a fraction f/g. Such a function corresponds to a function $V' \to k$ which is locally of the form $s_0^d F/G$ for some $d \in \mathbb{Z}$ where F, G are the homogeneous polynomials associated to f, g respectively.

Conversely, a regular function $V' \to k$ is locally of the form F/G for some homogeneous polynomials F, G of the same degree. Such a function corresponds to a function $V \to k$ which is locally of the form f/g for some polynomials f, g.

So we have a 1-1 correspondence between regular functions on V and V'. That is, we have an isomorphism. □

Let $X \subseteq \mathbb{P}^n_k$ be a quasi-projective algebraic set and $F_0 : \dots : F_m$ homogeneous polynomials of the same degree such that at least one of them is not identically zero on X. Then, we can define a well-defined function

$$(F_0 : \dots : F_m)\colon X - V(F_0, \dots, F_m) \to \mathbb{P}^m_k$$

by putting

$$(F_0 : \dots : F_m)(x_0 : \dots : x_n) = (F_0(x_0 : \dots : x_n) : \dots : F_m(x_0 : \dots : x_n))$$

Theorem 3.17. *Let $X \subseteq \mathbb{P}^n_k$ and $Y \subseteq \mathbb{P}^m_k$ be quasi-projective algebraic sets and $\phi\colon X \to Y$ a map (set theoretic). Then, ϕ is a regular map iff for every $x \in X$, there is a neighborhood $x \in U$, and homogeneous polynomials F_i of the same degree such that $\phi = (F_0 : \dots : F_m)$ on U.*

Proof. We take the rings $k[s_0, \dots, s_n]$ and $k[r_0, \dots, r_m]$ corresponding to $\mathbb{P}^n_k$ and $\mathbb{P}^m_k$, respectively.

First assume that ϕ is a regular map. Let $x \in X$ and V be the intersection of Y with one of the affine pieces in $\mathbb{P}^m_k$ which contains $\phi(x)$, say the one defined by $r_0 \neq 0$. Now r_i/r_0 is a regular function on V for any i, and by definition it gives a regular function ϕ_i on $\phi^{-1}V$. Moreover, the map $\phi^{-1}V \to V$ given by $(1 : \phi_1 : \dots : \phi_m)$ is identical to ϕ on $\phi^{-1}V$. Each ϕ_i is expressed as F_i/G_i on some neighborhood of x. If we take U to be the intersection of all these neighborhoods, then ϕ is given by $(1 : F_1/G_1 : \cdots : F_m/G_m)$ on U. Now $(G_1...G_m : F_1G_2 \dots G_m : \cdots)$ is the desired map.

Now assume that ϕ is a map which is locally given as in the statement of the theorem. First we prove that it is continuous. For each $x \in X$ there is a neighborhood $x \in U$ on which ϕ is given by $\pi = (F_0 : \cdots : F_m)$. It is enough to prove that $\pi = \phi|_U : U \to Y$ is continuous. A closed subset Z of Y is as $Z = V_Y(H_1, \dots, H_l)$. Now the composition of $H_1, \dots, H_l$ with π give the defining equations of $\pi^{-1}Z$ which is a closed subset of U.Hence, π is continuous.

Now let V be any open subset of Y and ψ a regular function on V. By definition, ψ is locally given as F/G. Hence the composition $\psi\phi$ is locally given by some F'/G' which means that $\psi\pi$ is a regular function on $\phi^{-1}V$. □

Corollary 3.18. *Let $X \subseteq \mathbb{P}^n_k$ be a quasi-projective algebraic variety and $Y \subseteq \mathbb{P}^m_k$ a quasi-projective algebraic set. A rational map $\pi \colon X \dashrightarrow Y$ is uniquely determined by some $\pi = (F_0 : \cdots : F_m)$ on some non-empty open subset $U \subseteq X$.*

Example 3.19. *Let $X = V(s_1s_2 - s_0^2) \subseteq \mathbb{P}^2_k$. Then, $\phi \colon X \to \mathbb{P}^1_k$ given by $\phi = (s_0 : s_1) = (s_2 : s_0)$ is a regular map. This is an isomorphism with the inverse $\theta \colon \mathbb{P}^1_k \to X$ given by $\theta = (r_0r_1 : r_1^2 : r_0^2)$. This example also shows that $X_1 = X - \{(0 : 1 : 0), (0 : 0 : 1)\}$ is isomorphic to $\mathbb{P}^1_k - \{(1 : 0), (0 : 1)\}$. The first one is a closed affine set in $\mathbb{A}^2_k$ but the second one is an open subset of $\mathbb{A}^1_k$.*

Example 3.20. *$\phi \colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ given by $\phi = (s_0^d, \dots, s_n^d)$ is a regular map. If $d > 1$, it is not an isomorphism because it is not 1-1.*

Example 3.21. *$\pi \colon \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ given by*

$$\pi = (s_1s_2, s_0s_2, s_0s_1) = (1/s_0, 1/s_1, 1/s_2)$$

is a rational map. It is not regular at the three points $(0 : 0 : 1)$, $(0 : 1 : 0)$, and $(1 : 0 : 0)$. However, it is birational its inverse being $\theta = (r_1r_2, r_0r_2, r_0r_1)$. This is called a Cremona transformation.

Example 3.22. *Let d be a natural number and $m = \left(\begin{array}{c} n+d \\ d \end{array} \right) - 1$. Define $\phi \colon \mathbb{P}^n_k \to \mathbb{P}^m_k$ by the m monomial $s_0^{d_0} \cdots s_n^{d_n}$ where $\sum d_i = d$ and $d_i \in \mathbb{N} \cup \{0\}$. Then, ϕ is a regular map which is called the Veronese embedding.*

Earlier we proved that any projective algebraic set is set-theoretically covered by affine algebraic sets.

Theorem 3.23. *Let X be a quasi-projective algebraic set. Then, for every $x \in X$, there is a neighborhood $x \in U$ which is affine, that is, it is isomorphic to an affine algebraic set.*

Proof. Fix $x \in X$. Since x is in one of the affine pieces covering the projective space, x has a neighborhood U which is quasi-affine. Now by replacing X with U we can assume that $X \subseteq \mathbb{A}^n_k$, that is, it is quasi-affine. Let $Y = \overline{U} = V(I)$ be the closure in $\mathbb{A}^n_k$. Further shrinking U, we can assume that $U = Y - V_Y(f)$ for some polynomial f. Now let $W = \mathbb{A}^n_k - V(f)$ and let Z be the affine algebraic set in $\mathbb{A}^{n+1}_k = \mathbb{A}^n_k \times \mathbb{A}^1_k$ defined by $Z = V(ft_{n+1} - 1)$. The map $\phi \colon Z \to W$ given by $\phi = (t_1, \dots, t_n)$ is a regular map. Moreover, the map $\psi \colon W \to Z$ given by $\psi = (t_1, \dots, t_n, 1/f)$ is the inverse of ϕ. So, Z and W are isomorphic and this induces an isomorphism between $U = W \cap Y$ and a closed subset of Z. □

According to this theorem, to study a quasi-projective algebraic set locally, it is enough to study affine algebraic sets.

4. Dimension

Let X be a topological space. The dimension of X, that is $\dim X$, is the sup of $l \in \mathbb{Z}$ such that there is a sequence $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_l$ of nonempty closed irreducible subsets. Dimension of a quasi-projective algebraic set is defined with respect to its Zariski topology.

Obviously, dimension of a single point is zero and dimension of $\mathbb{A}^1_k$ is 1. Dimension of $\emptyset$ by convention is -1.

Exercise 4.1. *Let X be a quasi-projective algebraic set. Prove that if $Y \subseteq X$ is a closed subset of X, then $\dim Y \leq \dim X$. Prove that if $X = X_1 \cup \cdots \cup X_l$ where X_i are closed subsets, then $\dim X = \max\{\dim X_i\}$.*

For any commutative ring R one can also define the (Krull) dimension as the sup of length of sequences of prime ideals $P_l \subsetneq \cdots \subsetneq P_0$. When X is an affine algebraic set, then $\dim X = \dim k[X]$ because irreducible closed subsets of X correspond to prime ideals of $k[X]$. We borrow the following theorem from commutative algebra and develop the dimension theory based on this theorem.

Theorem 4.2. *Let R be a finitely generated k-algebra integral domain. Let $0 \neq a \in R$ which is not a unit. Then, $\dim R/\langle a \rangle = \dim R - 1$.*

Corollary 4.3. *(i) Let $R = k[X]$ where X is an affine algebraic variety and let $f \in R$ such that $\emptyset \neq V_X(f) \neq X$. Then, $\dim V_X(f) = \dim X - 1$.*
(ii) $\dim \mathbb{A}^n_k = n$.

Proof. (i) $\dim k[V_X(f)] = \dim R/\sqrt{\langle f \rangle} = \dim R/\langle f \rangle = \dim R - 1$.
(ii) Let H be the affine algebraic variety defined by $t_n = 0$. Then, $\dim H = \dim \mathbb{A}^n_k - 1$. On the other hand, $H \simeq \mathbb{A}^{n-1}_k$, so by induction $\dim A^n_k = n$. □

Lemma 4.4. *Let X be an affine algebraic variety and U a non-empty open subset. Then, $\dim X = \dim U$.*

Proof. Let $Y = X - U$. It is enough to assume that $\emptyset \neq Y = V_X(g) \neq X$ for some g. We know that $\dim Y = \dim X - 1$. Now let f be a polynomial such that $V_X(f) \cap U \neq \emptyset$ but such that $V_X(f)$ does not contain any irreducible component of Y. Hence, $\dim V_X(f) = \dim X - 1 = \dim Z$ for some irreducible component Z of $V_X(f)$. This implies that Z is not inside Y, otherwise Y should have dimension more than $\dim X - 1$. Now by induction $\dim U > \dim Z \cap U = \dim Z = \dim X - 1$, therefore $\dim U = \dim X$. □

Theorem 4.5. *Let X be a quasi-projective variety. Then, $\dim X$ is the transcendence degree of $k(X)$ over k.*

Proof. Step 1. Let U be an affine neighborhood of a point of $x \in X$. Obviously, $\dim U \leq X$ and $\dim X$ is equal to the maximum of dimension of all such U. So, it is enough to prove the theorem for U and so we can assume that $X \subseteq \mathbb{A}^n_k$ is an affine variety.

Let d be the transcendence degree of $k(X)$ over k. Then, there are elements $z_1, \dots, z_d \in k(X)$ which are algebraically independent over k, and they generate a subfield $L = k(z_1, \dots, z_d) \subseteq k(X)$ such that $k(X)$ is a finite extension of L.

First suppose that $L = k(X)$. Then, using the z_i, we get a rational map $\pi \colon X \dashrightarrow \mathbb{A}^d_k$ given by $\pi = (z_1, \dots, z_d)$. Since we can recover all the variables t_i on X, π is birational. On the other hand, $\mathbb{A}^d_k$ can be considered as a hypersurface Y in $\mathbb{A}^{d+1}_k$.

Now assume that $L \neq k(X)$ for any set $z_1, \dots, z_d$ of algebraically independent elements. Then by [4, Appendix 5, Proposition 1], we can find algebraically independent elements $z_1, \dots, z_d$ and an element $z_{d+1} \in k(X) - L$ such that $k(X) = L(z_{d+1})$ and $k(X)$ is a separable extension of L. There is an irreducible polynomial $f \in k[s_1, \dots, s_{d+1}]$ such that $f(z_1, \dots, z_{d+1}) = 0$. Define $Y = V(f)$ in the affine space $\mathbb{A}^{d+1}_k$. The function field $k(Y) = k(X)$. So, again X is birational to a hypersurface in $\mathbb{A}^{d+1}_k$.

Step 2. Since X and Y are birational, they have isomorphic open subsets U and V. Therefore, $\dim X = \dim Y$.

Step 3. Since Y is a hypersurface in $\mathbb{A}^{d+1}_k$, its dimension is d. $\square$

In particular, we also proved the following

Corollary 4.6. *Any quasi-projective algebraic variety of dimension d is birational to an irreducible hypersurface in $\mathbb{A}^{d+1}_k$.*

Exercise 4.7. *Let X be a quasi-projective variety and U a dense open subset. Prove that $\dim U = \dim X$. In particular, $\dim \mathbb{P}^n_k = n$.*

Theorem 4.8. *Any two projective curves in $\mathbb{P}^2_k$ intersect.*

Proof. Let $X = V(F)$ and $Y = V(G)$ be two curves in $\mathbb{P}^2_k$. Since X, Y are curves they are irreducible by definition so F, G are irreducible and $\langle F \rangle \neq \langle G \rangle$ in $R = k[s_0, s_1, s_2]$. However, the theorem is true even when X, Y are not irreducible.

Let $Z = V(F, G) \subseteq \mathbb{A}^3_k$. Note that since F and G are homogeneous, Z contains the origin and so it is not empty. We prove that $\dim Z = 1$. Let $R' = R/\langle F \rangle$ and $R'' = R'/\langle G \rangle$. By previous theorems, $\dim R = 3$, $\dim R' = 2$ and finally $\dim R'' = 1$. On the other hand, $R'' \simeq R/\langle F, G \rangle$ and $\dim R'' = \dim Z$, so $\dim Z = 1$.

Now any point of Z which is not the origin gives a point of $X \cap Y$, therefore $X \cap Y \neq \emptyset$. $\square$

Lemma 4.9. *Let X be an affine algebraic variety and let $f \in k[X]$ such that $\emptyset \neq V_X(f) \neq X$. Then, every irreducible component of $V_X(f)$ has dimension* $\dim X - 1$.

Proof. Let Z be an irreducible component of $V_X(f)$. Let U be an affine open subset of X which intersects Z but not the other components of $V_X(f)$. Now f is also a regular function on U and $V_U(f) = Z \cap U$. So, dimension of $Z \cap U$ is $\dim U - 1 = \dim X - 1$. Thus, $\dim Z = \dim X - 1$. $\square$

For a regular map $\phi\colon X \to Y$, we define the fibre of ϕ over $y \in Y$ to be $X_y := \phi^{-1}\{y\}$.

Theorem 4.10. *Let $\phi\colon X \to Y$ be a surjective regular map of quasi-projective algebraic varieties,* $\dim X = d$ *and* $\dim Y = d'$. *Then,*
(i) for any $y \in Y$, $\dim X_y \geq d - d'$,
(ii) $B \subseteq Y$, the set of points $y \in Y$ such that $\dim X_y = d - d'$, *is dense in Y.*

Proof. (i) Fix $y \in Y$. By replacing Y with an affine neighborhood of y, we can assume that Y is affine. We construct sequences $Z_{d'} \subsetneq Z_{d'-1} \subsetneq \cdots \subsetneq Z_0 = Y$ and $S_{d'} \subsetneq S_{d'-1} \subsetneq \cdots \subsetneq S_0 = X$ of closed subsets such that $S_i = \phi^{-1} Z_i$ and every irreducible component of Z_i has dimension $d' - i$ and every irreducible component of S_i has dimension $\geq d - i$.

Suppose that we have constructed the sequence up to $i - 1 < d'$. We construct Z_i and S_i as follows. For each irreducible component of Z_{i-1} choose a point distinct from y and call the set of these points M. Now let h be a regular function on Y such that $h(y) = 0$ but $h(y') \neq 0$ if $y' \in M$. Define $Z_i = V_{Z_{i-1}}(h)$. Then, every irreducible component of Z_i has dimension $d' - i$ and every irreducible component of $S_i = \phi^{-1} Z_i$ has dimension $\geq d - i$.

By construction $\dim Z_{d'} = 0$ and $y \in Z_{d'}$. That is, $Z_{d'}$ is a finite set containing y. Every component of X_y is also a component of $S_{d'}$. Therefore, $\dim X_y \geq d - d'$.

(ii) We proceed similar to (i). First, we prove that $B \neq \emptyset$. We construct sequences $Z_{d'} \subsetneq Z_{d'-1} \subsetneq \cdots \subsetneq Z_0 = Y$ and $S_{d'} \subsetneq S_{d'-1} \subsetneq \cdots \subsetneq S_0 = X$ of closed subsets such that $S_i = \phi^{-1} Z_i$ and every irreducible component of Z_i has dimension $d' - i$ and every irreducible component of S_i has dimension $d - i$.

Suppose that we have constructed the sequence up to $i - 1 < d'$. We construct Z_i and S_i as follows. For each irreducible component of

Z_{i-1} choose a point and call the set of these points M. Moreover, for each irreducible component T of S_{i-1} choose a point in $\phi(T)$ and add it to M. Now let h be a regular function on Y such that $V_{Z_{i-1}}(h) \neq \emptyset$ but $h(y) \neq 0$ if $y \in M$. Define $Z_i = V_{Z_{i-1}}(h)$. Then, every irreducible component of Z_i has dimension $d' - i$ and every irreducible component of $S_i = \phi^{-1} Z_i$ has dimension $d - i$.

By construction $\dim Z_{d'} = 0$, that is, $Z_{d'}$ is a finite set. Every irreducible component of $S_{d'}$ has dimension $d - d'$, so for some $y \in Y$, $\dim X_y = d - d'$.

Now assume that $\overline{B} \neq Y$, then by taking an affine open subset $W \subset Y - \overline{B}$ and considering the induced regular map $\phi^{-1} W \to W$ we get a contradiction. Therefore, B is dense in Y. $\square$

5. Local properties and smoothness

Let R be a commutative ring and P a prime ideal of R. Then, R_P is a local ring which is defined as

$$R_P := \{\frac{f}{g} \mid f, g \in R, g \notin P\}$$

where by convention $\frac{f}{g} = \frac{f'}{g'}$ iff there is $h \notin P$ such that $h(fg' - f'g) = 0$. There is a natural R-homomorphism $R \to R_P$ sending f to the class of $\frac{f}{1}$, and for each ideal I of R the ideal I_P in R_P is generated by I, and every ideal of R_P is of this form. In particular, P_P is the maximal ideal of R_P. If R is noetherian, then so is R_P.

Since any point on a quasi-projective algebraic set has an affine neighborhood, we restrict ourselves to affine algebraic sets when studying local properties.

Definition 5.1. *Let X be an affine algebraic set, $R = k[X]$ its coordinate ring, and Z an irreducible closed subset of X. The ideal of Z in R, that is $P = I_Z$ is a prime ideal and we call R_P the local ring of X at Z and we denote it by $\mathcal{O}_{X,Z}$.*

Dimension of a quasi-projective algebraic set X at $x \in X$ is defined as the maximum of dimension of irreducible components containing x. It is denoted by $\dim_x X$.

Lemma 5.2. *Let X be an affine algebraic set and $x \in X$. Then,*
(i) elements of $\mathcal{O}_{X,x}$ can be considered as limits of functions which are regular at x,
(ii) if U is an affine neighborhood of x, then $\mathcal{O}_{U,x} \simeq \mathcal{O}_{X,x}$,
(iii) $\dim \mathcal{O}_{X,x} = \dim_x X$.

Proof. (i) Let ψ and ρ be regular functions on the neighborhoods U and V of x, respectively. We say that (ψ, U) and (ρ, V) are equivalent if there is a neighborhood $W \subseteq U \cap V$ of x on which ψ and ρ are equal. Let S be the ring of the equivalence classes. Define a homomorphism $r \colon \mathcal{O}_{X,x} \to S$ by sending f/g to $(\psi = f/g, U)$ where g has no zero on $x \in U$. This is well-defined because if $f/g = f'/g'$, then f/g and f'/g' are equal on a neighborhood of x since there is $h \notin I_x$ such that $h(fg' - f'g) = 0$. If $r(f/g) = 0 \in S$, then f should be zero on a neighborhood of x which implies that there $h \notin I_x$ such that $hf = 0$ on X and so $f/g = 0 \in \mathcal{O}_{X,x}$, so r is injective. Now let $(\psi, U) \in S$. By definition, there are $f, g \in k[X]$ and a neighborhood $x \in V$ such that $\psi = f/g$ on V. Thus, $(\psi, V) = r(f/g)$. Therefore r is an isomorphism of rings.

(ii) S in (i) is the same if defined on X or U.

(iii) Let $d = \dim_x X$. By replacing X with some affine neighborhood of x, we may assume that $\dim_x X = \dim X$. Moreover, we have $Z_0 = \{x\} \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d$ a sequence of irreducible closed subsets of X by the proof of Theorem 4.10. This corresponds to a sequence of prime ideals $I_{Z_d} \subsetneq \cdots \subsetneq I_{Z_0}$ in $k[X]$ which in turn corresponds to a sequence of prime ideals in $\mathcal{O}_{X,x}$ of the same length. So, $d \le \dim \mathcal{O}_{X,x}$.

Conversely, any maximal sequence of prime ideals in $\mathcal{O}_{X,x}$ correspond to a sequence of prime ideals in $k[X]$ which in turn corresponds to a sequence of irreducible closed subsets of X the smallest one being $Z_0 = \{x\}$.

□

Definition 5.3. *Let $X \subseteq \mathbb{A}^n_k$ be an affine algebraic set, $x = (x_1, \dots, x_n) \in X$ and $I_X = \langle f_1, \dots, f_m \rangle$. The tangent space $T_{X,x}$ of X at x is given by the following differential equations*

$$d_x f_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial t_j}(x)(t_j - x_j) = 0$$

So, $T_{X,x}$ is a linear subspace of $\mathbb{A}^n_k$ and so a vector space over k with the origin at x. Moreover, $T_{X,x}$ is independent of the choice of the generators f_i.

Exercise 5.4. *Let $g, h \in k[t_1, \dots, t_n]$ and $x \in \mathbb{A}^n_k$. Prove that $d_x(g + h) = d_x g + d_x h$ and $d_x(gh) = g(x) d_x h + h(x) d_x g$.*

If $x = 0 \in X$, then the equations of the tangent space $T_{X,x}$ are given by the linear part of the f_i. The tangent space can also be defined as the union of all lines tanget to X at x. A line $L \subset \mathbb{A}^n_k$ through the origin 0 is determined by one other point $0 \neq B \in \mathbb{A}^n_k$, that is, $L = \{aB \mid a \in k\}$. A line $L \subset \mathbb{A}^n_k$ is tangent to X at x if $f_i(aB)$ is a multiple of a^2 considered as a polynomial in a, for all i. This is equivalent to saying that g_i the linear part of f_i vanishes on every point of L, then $T_{X,x} = V(g_1, \dots, g_m)$.

Example 5.5. *Let $X = \mathbb{A}^n_k$. Then, $T_{X,x} = \mathbb{A}^n_k$ for every x.*

Example 5.6. *Let $X \subsetneq \mathbb{A}^n_k$ with $I_X = \langle f \rangle$. Then, $T_{X,x}$ is defined by the single equation $d_x f = 0$. Thus, $T_{X,x} = \mathbb{A}^n_k$ iff $\frac{\partial f}{\partial t_j}(x) = 0$ for all j. On the other hand, if some $\frac{\partial f}{\partial t_j}(x) \neq 0$, then $\dim_k T_{X,x} = n - 1$.*

Note that since $I_X = \langle f \rangle$, we have a decomposition $f = f_1 \cdots f_m$ where the f_i are distinct irreducible polynomials. Therefore, $\mathrm{d}_x f$ cannot be zero for every x. This would not be true if we did not assume $I_X = \langle f \rangle$.

Suppose that the characteristic of k is $p > 0$. Then for $g = t_1^p$ we have $\frac{\partial g}{\partial t_j} = 0$ for all j. More generally, a polynomial g satisfies $\frac{\partial g}{\partial t_j} = 0$ for every j iff g is a polynomial in $t_1^p, \dots, t_n^p$. In particular, a nonzero g with the latter property is not irreducible because $(e+h)^p = e^p + h^p$ for any two polynomials e, h.

Theorem 5.7. *Let $X \subseteq \mathbb{A}_k^n$ be an affine algebraic set. Then, the dual $T_{X,x}^*$ is isomorphic to I_x/I_x^2 as vector spaces where I_x is the ideal of x in $k[X]$.*

Proof. First note that I_x/I_x^2 is a module over $k[X]/I_x \simeq k$ hence a k-vector space.

Let $\phi \in k[X]$ which is given by $\phi = g$ for some $g \in k[t_1, \dots, t_n]$. The differential $d_x g$ is a linear function on $\mathbb{A}_k^n$ and so on $T_{X,x}$. However, if $\phi = g'$ for another g', then $d_x g = d_x g'$ may not hold on $\mathbb{A}_k^n$. But, the equality holds on $T_{X,x}$ because $g - g' \in I_X$ the ideal of X in $k[t_1, \dots, t_n]$. So, we can define $d_x \phi = d_x g$. Therefore, we have a well-defined surjective map $d_x \colon I_x \to T_{X,x}^*$.

On the other hand, if $\phi \in I_x^2$, then $d_x \phi = 0$ on $T_{X,x}$. This is the case because if $g, h \in I_x$, then $d_x(gh) = g(x) d_x h + h(x) d_x g$ which shows that $d_x(gh)$ vanishes on $T_{X,x}$. The ideal I_x^2 is generated by such gh so this proves the claim.

It is enough to prove that the kernel of d_x is I_x^2. Let $\phi = g \in I_x$ such that $d_x \phi = d_x g = 0$ on $T_{X,x}$. Suppose that $I_X = \langle f_1, \dots, f_m \rangle$ and so $T_{X,x} = V(d_x f_1, \dots, d_x f_m)$. Since $g - d_x g \in I_x^2$, by replacing g with $d_x g$ we can assume that $d_x g = g$. Since $g = d_x g \in I_{T_{X,x}}$, we can write $g = \sum_{i=1}^m a_i d_x f_i$ where $a_i \in k$. This implies that $g - \sum_{i=1}^m a_i f_i = \sum_{i=1}^m a_i (d_x f_i - f_i) \in I_x^2$. Hence, $g \in I_x^2$. Therefore the kernel of d_x is exactly I_x^2 and this induces an isomorphism between I_x/I_x^2 and $T_{X,x}^*$ as vector spaces over k. □

Let $\phi \colon X \to Y$ be a regular map of affine algebraic sets and $y = \phi(x)$. The dual homomorphism $\phi^* \colon k[Y] \to k[X]$ induces a homomorphism $I_y/I_y^2 \to I_x/I_x^2$ because $\phi^*(I_y) \subseteq I_x$ and $\phi^*(I_y^2) \subseteq I_x^2$. This in turn induces a k-homomorphism $d_x \phi \colon T_{X,x} \to T_{Y,y}$ of vector spaces which is called the differential of ϕ at x.

Corollary 5.8. *Let $X \subseteq \mathbb{A}_k^n$ be an affine algebraic set. Then, $T_{X,x}$ depends only on the isomorphism class of X.*

Proof. If $\phi \colon X \to Y$ is an isomorphism of affine algebraic sets and $y = \phi(x)$. Then, $\phi^* \colon k[Y] \to k[X]$ is also an isomorphism and gives an isomorphism between I_x/I_x^2 and I_y/I_y^2 which proves that $T_{X,x} \simeq T_{Y,y}$. □

Exercise 5.9. *Define a differential map* $d_x\colon \mathcal{O}_{X,x} \to T^*_{X,x}$ *by*

$$d_x(f/g) = \frac{g(x)d_xf - f(x)d_xg}{g(x)^2}$$

Prove that this is well-defined and it gives an isomorphism $m_x/m_x^2 \to T^*_{X,x}$ *as* k*-vector spaces where* m_x *is the maximal ideal. (Hint: use the fact that the differential map* $d_x\colon I_x/I_x^2 \to T^*_{X,x}$ *defined before factors through* $I_x/I_x^2 \to m_x/m_x^2 \to T^*_{X,x}$*)*

Definition 5.10. *Let* X *be a quasi-projective algebraic set and* $x \in X$*. Define the tangent space* $T_{X,x}$ *to be* $T_{U,x}$ *for an affine neighborhood* $x \in U$*.*

Definition 5.11. *Let* X *be a quasi-projective algebraic set and* $x \in X$*. We say that* X *is smooth or nonsingular at* x *if* $\dim_k T_{X,x} = \dim_x X$*. A point* x *which is not smooth is called singular. We say that* X *is smooth or nonsingular if all of its points are.*

Lemma 5.12. *Let* X *be a quasi-projective algebraic variety. Then,* $\dim_k T_{X,x} \geq \dim X$ *for any* $x \in X$*. Moreover, the equality holds on some dense open subset.*

Proof. By replacing X with an affine neighborhood of x, we may assume that $X \subseteq \mathbb{A}^n_k$ is affine and put $d = \dim X$. By Corollary 4.6, X is birational to a hypersurface $Y = V(f) \subset \mathbb{A}^{d+1}_k$ with f irreducible. For a point $y \in Y$, the tangent space $T_{Y,y}$ is defined by the single linear equation $d_yf = 0$. The point $y \in Y$ is singular iff the equation d_yf is identically zero on $\mathbb{A}^{d+1}_k$ iff $\frac{\partial f}{\partial t_j}(y) = 0$ for all $1 \leq j \leq d+1$. Therefore, the set of singular y is a proper closed subset of Y (see Example 5.6). In other words, there is an open subset of Y for whose points the tangent space has dimension d. Since X and Y are birational, this implies that there is an open subset of X for whose points the tangent space has dimension d, that is, they are smooth points.

Now let $S \subset X \times \mathbb{A}^n_k$ be defined as

$$S = \{(x,a) \in X \times \mathbb{A}^n_k \mid a \in T_{X,x}\}$$

S is a closed subset of $X \times \mathbb{A}^n_k$ and we have the natural regular map $\phi\colon S \to X$ given by $\phi(x,a) = x$. Clearly, the fibre S_x is isomorphic to the tangent space $T_{X,x}$, and hence every fibre is irreducible. Moreover, by sending x to (x,x) we can embed X into S, that is, X is isomorphic to $X' = \{(x,x) \mid x \in X\} \subseteq S$. Let S' be an irreducible component of S which contains $T_{X,x}$ for a dense subset Z of points x. Then, $X' \subseteq S'$ since $X' \cap S'$ is closed and dense in X'. So, the induced map $S' \to X$ is surjective and we call it ψ.

By the theorem on the dimension of fibres, $\dim T_{X,x} \geq \dim S' - \dim X$. As we already saw, there is a dense open subset $U \subseteq X$ such that X is smooth at each point of U. Let $x = (x_1, \cdots, x_n) \in U \cap Z$ and we may assume that $u_1 = t_1 - x_1, \ldots, u_d = t_d - x_d$ are local parameters at $x = (x_1, \cdots, x_n)$ (see Definition 5.22 and the subsequent lemma). Then, in a neighborhood of x, we have $\dim \bigcap_{i=1}^{l} V_X(u_i) = d - l$ by Lemma 4.9 and $x = \bigcap_{i=1}^{d} V_X(u_i)$. Accordingly, in a nieghborhood of the fibre S'_x, we have $\dim \bigcap_{i=1}^{d} \psi^{-1} V_X(u_i) = d$ and $\dim \bigcap_{i=1}^{l} \psi^{-1} V_X(u_i) \geq 2d - l$. Therefore, $\dim S' = 2d$.

Now by applying the theorem on dimension of fibres we get $\dim T_{X,x} \geq \dim S' - \dim X = \dim X$. □

Exercise 5.13. *Find the mistake in the proof of the previous lemma.*

Lemma 5.14. *Let X be a quasi-projective algebraic set. Then,* $\dim T_{X,x} \geq \dim_x X$ *for any $x \in X$. Moreover, the equality holds on a dense open subset.*

Proof. It is enough to prove that $\dim T_{X,x} \geq \dim_x X$. As usual, by replacing X, we may assume that X is affine and that $\dim_x X = \dim X$. Let Z be an irreducible component of X containing x with maximal dimension. From the definition of tangent spaces, it is clear that $T_{Z,x} \subseteq T_{X,x}$. Since $\dim T_{Z,x} \geq \dim Z = \dim_x X$, then $\dim T_{X,x} \geq \dim_x X$. □

Corollary 5.15. *Let X be a quasi-projective algebraic set. Then, the set of smooth points of X contains a dense open subset.*

With a bit more work, one can prove that the set of smooth points of a quasi-projective algebraic variety is actually open. To prove this one should prove that the set B in Theorem 4.10 is open. Later we prove that the points in the intersection of two irreducible components of a quasi-projective algebraic set is singular so again the set of smooth points should be open.

Exercise 5.16. *Let X and Y be affine algebraic sets, $x \in X$ and $y \in Y$. Prove that if x and y are smooth points, then (x, y) is a smooth point of $X \times Y$.*

Example 5.17. *Let $X \subset \mathbb{A}_k^2$ be defined by $f = t_2^2 - t_1^3$. Then, $0 = (0, 0)$ is a singular point because $T_{X,0} = \mathbb{A}_k^2$. Earlier we proved that X is birational to the affine line $\mathbb{A}_k^1$ but not isomorphic. The presence of the singular point 0 prevents an isomorphism.*

Example 5.18. *Suppose that the characteristic of k is not 2 and let $X \subset \mathbb{P}_k^n$ be defined by $F = s_0^2 + \cdots + s_m^2 = 0$ where $m \leq n$. If $m = n$,* *then X has no singular points. But if $m < n$, then all the points $x = (x_0 : \cdots : x_n) \in X$ with $x_0 = \cdots = x_m = 0$ are singular.*

Example 5.19. *Let $X \subset \mathbb{A}^2_k$ be defined by $f = t_2^2 - t_1^3 - t_1^2$. Then, $0 = (0,0)$ is a singular point because $T_{X,0} = \mathbb{A}^2_k$. At the origin, X has two branches and the part of f with smallest degree, that is, $t_2^2 - t_1^2$ gives the equations of the tangent lines to these branches.*

Example 5.20. *An elliptic curve $X \subset \mathbb{P}^2_k$ is a smooth curve given by F of degree 3. For example, $F = t_2^2 t_0 - t_1^3 + t_1 t_0^2$ defines an elliptic curve.*

Exercise 5.21. *Find the singular points of*

$$X = V(t_1^2 t_2^2 + t_0^2 t_2^2 + t_0^2 t_1^2 - t_0 t_1 t_2 t_3) \subset \mathbb{P}^3_k$$

Definition 5.22. *Let X be a quasi-projective algebraic set and $x \in X$ a smooth point with $\dim_x X = d$. Elements $u_1, \ldots, u_d \in m_x \subset \mathcal{O}_{X,x}$ are called local parameters if their images form a basis of m_x/m_x^2 as a vector space over k where m_x is the maximal ideal of $\mathcal{O}_{X,x}$.*

Lemma 5.23. *Let X be a quasi-projective algebraic set and $x \in X$ a smooth point and $u_1, \ldots, u_d \in m_x \subset \mathcal{O}_{X,x}$ local parameters. Then, $u_1, \ldots, u_d$ generate the maximal ideal m_x.*

Proof. Let $J \subseteq m_x$ be the ideal generated by the local parameters and $M = m_x/J$. Since the local parameters form a basis of m_x/m_x^2, $J + m_x^2 = m_x$. So, $m_x M = (m_x^2 + J)/J = m_x/J = M$. Now by Nakayama's lemma, $M = 0$, hence $J = m_x$. □

Definition 5.24. *Let X be a quasi-projective algebraic set and $x \in X$ a smooth point with $u_1, \ldots, u_d \in m_x \subset \mathcal{O}_{X,x}$ local parameters. We say that $f \in \mathcal{O}_{X,x}$ has an associated formal power series*

$$\Lambda = \lambda_0 + \lambda_1 + \cdots \in k[[s_1, \ldots, s_d]]$$

if $f - \lambda'_0 - \lambda'_1 - \cdots - \lambda'_l \in m_x^{l+1}$ where $\lambda'_i = \lambda_i(u_1, \ldots, u_d)$. Here λ_i is the homogeneous part of Λ of degree i.

Lemma 5.25. *Let X be a quasi-projective algebraic set and $x \in X$ a smooth point with $u_1, \ldots, u_d \in m_x \subset \mathcal{O}_{X,x}$ local parameters. Then, any $f \in \mathcal{O}_{X,x}$ has an associated formal power series.*

Proof. Put $\lambda'_0 = f(x)$. So, $f - \lambda'_0 \in m_x$ and we can find $a_1, \ldots, a_d \in k$ such that $f - \lambda'_0 - \lambda'_1 \in m_x^2$ where $\lambda'_1 = \sum a_i u_i$. On the other hand, we can write $f - \lambda'_0 - \lambda'_1 = \sum g_j h_j$ where $g_j, h_j \in m_x$. Similarly, each g_j and h_j can also be written as the sum of a homogeneous polynomial of degree 1 in $k[u_1, \ldots, u_d]$ and an element in m_x^2. So, we can find a homogeneous polynomial λ_2' in $k[u_1, \ldots, u_d]$ of degree 2 (note that $\lambda_2' = 0$ is also allowed) such that $f - \lambda_0' - \lambda_1' - \lambda_2' \in m_x^3$. By continuing this process we can find $\lambda_1', \ldots, \lambda_l'$ such that $f - \lambda_0' - \lambda_1' - \cdots - \lambda_l' \in m_x^{l+1}$. In this way, we get a corresponding formal power series $\Lambda = \lambda_0 + \lambda_1 + \cdots \in k[[s_1, \ldots, s_d]]$. □

Theorem 5.26. *Let X be a quasi-projective algebraic set and $x \in X$ a smooth point and $u_1, \ldots, u_d \in m_x \subset \mathcal{O}_{X,x}$ local parameters. Then, the formal power series Λ associated to $f \in \mathcal{O}_{X,x}$ is unique. This gives an embedding of $\mathcal{O}_{X,x}$ into $k[[s_1, \ldots, s_d]]$. Moreover, since the latter ring is a UFD, the local ring $\mathcal{O}_{X,x}$ is also a UFD.*

For an elementary proof see [4,II.§2, Theorem 4, and §3, Theorem 2]. A more advanced explanation is that the completion $\hat{\mathcal{O}}_{X,x}$ is the ring $k[[s_1, \ldots, s_d]]$ and associating formal power series to elements of $\mathcal{O}_{X,x}$ gives an embedding of $\mathcal{O}_{X,x}$ into $\hat{\mathcal{O}}_{X,x} = k[[s_1, \ldots, s_d]]$.

Lemma 5.27. *Let X be a quasi-projective algebraic set and $x \in X$ a smooth point. Then, only one irreducible component of X passes through x.*

Proof. By replacing X with a suitable affine neighborhood of x we can assume that $k[X] \subset \mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is isomorphic to a subring of $k[[s_1, \ldots, s_d]]$, it is an integral domain. Thus, $k[X]$ should also be an integral domain which in turn implies that X is irreducible. □

Exercise 5.28. *Let $X = \mathbb{A}_k^1$, $x \in X$ and $f \in k[t]$. Prove that f is a local parameter at x iff x is a simple root of f, i.e. $t - x$ divides f but $(t - x)^2$ does not.*

Exercise 5.29. *Let $x \in X$ be a singular point. Prove that any $f \in \mathcal{O}_{X,x}$ has infinitely many different associated formal power series.*

Proof. We may replace X by an affine neighborhood such that every component passes through x and $k[X] \subset \mathcal{O}_{X,x}$.

First assume that X is irreducible. Let $u_1, \ldots, u_d \in k[X]$ such that they form a basis of m_x/m_x^2. Since x is a singular point, $d > \dim X$. Thus, $u_1, \ldots, u_d$ are algebraically dependent over k and $h(u_1, \ldots, u_d) = 0$ for some nonzero polynomial $h \in k[s_1, \ldots, s_d]$. Now $\Lambda = h$ is a power series associated to 0. This proves that we can associate infinitely many power series to 0 and so to any f.

Now assume that X is not irreducible and $X = X_1 \cup \cdots \cup X_n$ is the decomposition of X into irreducible components. As above, we can find $h_i \in k[s_1, \ldots, s_d]$ such that $h_i(u_1, \ldots, u_d) = 0$ on X_i. Put $h = h_1 \cdots h_n$. The rest is similar. □

Definition 5.30. *Let X be a quasi-projective algebraic set, $x \in X$ and $Y \subseteq X$ a closed subset. The ideal m_Y in $\mathcal{O}_{X,x}$ is the set of elements vanishing on Y in some neighborhood of x. Equivalently, if $x \in U$ is an affine neighborhood, m_Y is the extension of the ideal I_Y in $k[U]$.*

Theorem 5.31. *Let X be a smooth quasi-projective algebraic variety of dimension d and $Y \subset X$ a subvariety of dimension $d-1$. Then, the ideal m_Y in $\mathcal{O}_{X,x}$ is principal for any $x \in X$. This in particular means that for any $x \in X$ there is a neighborhood $x \in U$ and a regular function f on U such that $I_Y = \langle f \rangle$ in $k[U]$.*

Proof. Let $x \in X$. Replacing X, we can assume that it is an affine variety and $x \in Y$. Let $f \neq 0$ be an irreducible element of m_Y. Obviously, $\langle f \rangle \subseteq m_Y$.

On some affine neighborhood $x \in U$ we have $V_U(f) = (Y \cap U) \cup Z$ where Z is the union of the irreducible components of $V_U(f)$ other than $Y \cap U$. If $x \notin Z$, then by shrinking U, $V_U(f) = Y \cap U$. If $x \in Z$, let g and h be regular functions on U such that g vanishes on $Y \cap U$ but not on Z and such that h vanishes on Z but not on $Y \cap U$. The regular function gh vanishes on $V_U(f)$. Thus, gh belongs to the ideal $\sqrt{\langle f \rangle} = \langle f \rangle \subset \mathcal{O}_{X,x}$ and since f is irreducible, g or h belongs to $\langle f \rangle$. Suppose that $g \in \langle f \rangle$ which means that $g = fe$ for some $e \in \mathcal{O}_{X,x}$. By shrinking U such that e becomes regular on U, we get a contradiction because $V_U(f) \subseteq V_U(g = fe)$. Therefore, we can choose U such that $V_U(f) = Y \cap U$.

Now let $c \in m_Y$ be any element. Then, on some affine neighborhood $x \in W \subseteq U$ we have $V_W(f) \subseteq V_W(c)$ such that f and c are regular on W. So, $c \in \sqrt{\langle f \rangle} = \langle f \rangle \subset \mathcal{O}_{X,x}$. Therefore, $m_Y = \langle f \rangle$. □

Theorem 5.32. *Let $\pi \colon X \dashrightarrow Y$ be a rational map where X is a smooth quasi-projective variety of dimension d and Y is a projective algebraic set. Then, the set of points where π is not regular is of dimension $\leq d-2$.*

Proof. Let $Z \subset X$ be the set of points where π is not regular. We know that Z is a closed subset. Suppose that $\dim Z = d-1$ and let S be an irreducible component of Z with dimension $d-1$. Let $x \in S$.

Assume that $Y \subseteq \mathbb{P}^m_k$. After replacing X with an affine neighborhood of x, π is given by $\pi = (f_0 : \cdots : f_m)$ where $f_i \in k[X]$. We can assume that the f_i do not have a common factor in $\mathcal{O}_{X,x}$. Since π is not regular at any point of S, $f_i(S) = 0$ for all i. On the other hand, there is $f \in \mathcal{O}_{X,x}$ such that $m_S = \langle f \rangle$. Therefore, f divides each f_i, that is, we can write $f_i = g_i f$ for some $g_i \in \mathcal{O}_{X,x}$. This is a contradiction. □

Corollary 5.33. *Let X and Y be birational smooth projective curves. Then, they are isomorphic.*

Proof. Let $\pi\colon X \dashrightarrow Y$ be a birational isomorphism. By the previous theorem, π is a regular map and for the same reason its inverse is also a regular map and hence an isomorphism. $\square$

6. Finite maps and normal varieties

One of the most important problems in algebraic geometry is to prove that any quasi-projective variety is birational to a smooth projective variety. This is still open when the characteristic of k is positive. Though this problem is far beyond this lecture but it is not difficult to prove a very weak version of it. That is, one can prove that any quasi-projective variety is birational to a normal projective variety. In dimension one, normality and smoothness are equivalent conditions.

Remember that if R is a commutative ring and S an R-algebra, the integral closure of R in S is also an R-algebra. If S is a finitely generated R-algebra, then S is integral over R iff it is a finite R-module.

Definition 6.1. *Let $\phi\colon X \to Y$ be a regular map of quasi-projective algebraic sets. We say that ϕ is finite if for every $y \in Y$, there is an affine neighborhood $y \in W$ such that $U = \phi^{-1}W$ is also affine and $k[U]$ is a finite $k[W]$-module.*

Roughly speaking, a finite map is a proper regular map whose fibres are all finite sets (see [2, II, §4] for definition of proper maps). When k is the field of complex numbers, properness has a simple meaning. In this case, a regular map is proper if and only if the inverse image of every compact set is compact in the sense of usual complex topology (not the Zariski topology).

Exercise 6.2. *Let $\phi\colon \mathbb{A}^2_k \to \mathbb{A}^1_k$ be the regular map which sends $(x_1, x_2) \in \mathbb{A}^2_k$ to $x_1 \in \mathbb{A}^1_k$. Prove that ϕ is not a finite map.*

Exercise 6.3. *Let $\phi\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ be the regular map which sends $(x_0 : x_1)$ to $(x_0^l : x_1^l)$ for a fixed $l \in \mathbb{N}$. Prove that ϕ is a finite map.*

Let R be an integral domain and K its fraction field. R is called normal if it is integrally closed in K. It is known that R is normal iff R_P is normal for all prime ideals P of R iff R_m is normal for all maximal ideals m. Any UFD is normal. Moreover, if R is a local noetherian ring of dimension one, then R is normal iff its maximal ideal is principal. In this case, R is a DVR.

Definition 6.4. *Let X be a quasi-projective algebraic set. We say that X is normal if all the local rings $\mathcal{O}_{X,x}$ are normal. Let Y be a quasi-projective algebraic variety. If we have a surjective finite birational regular map $\phi\colon X \to Y$ where X is a normal quasi-projective variety, we say that X or ϕ is a normalisation of Y.*

For an affine algebraic variety X, being normal means that every finite birational regular map $\phi\colon X' \to X$ is an isomorphism.

Note that the irreducible components of a normal quasi-projective algebraic set are disjoint because the local rings are integral domains.

Exercise 6.5. *Let X be a smooth quasi-projective algebraic set. Prove that it is normal.*

Exercise 6.6. *Let X be an affine algebraic variety of dimension one. Prove that if X is normal, then it is smooth.*

Theorem 6.7. *Let X be a normal quasi-projective algebraic variety of dimension d. Then, the set of singular points of X is contained in a closed subset of dimension $\leq d-2$.*

Proof. We can assume that X is affine. We already know that the set of singular points of X is not dense. Let Z be its closure. Assume that Z has a component T of dimension $d-1$ and let $P = I_T$ be its prime ideal.

Since X is normal, R_P is normal where $R = k[X]$. Moreover, R_P is a local ring of dimension one because $\dim T = \dim X - 1$. Thus, its maximal ideal is generated by one element. Hence, replacing X by a smaller affine open subset, we can assume that P is principal, say $P = \langle u \rangle$. Note also that $R' := k[T] = R/P$.

There is a dense open subset $W \subseteq T$ on which T is smooth. Let $x \in W$. So, there are local parameters $u_1, \ldots, u_{d-1} \in R'$ at $x \in T$. Let $v_1, \ldots, v_{d-1} \in R$ be such that the image of v_i is u_i under the natural homomorphism $R \to R'$. Let $m_{T,x}$ and $m_{X,x}$ be the maximal ideals of x in $\mathcal{O}_{T,x}$ and $\mathcal{O}_{X,x}$ respectively. In fact, $m_{X,x}$ is the inverse of $m_{T,x}$ under the natural homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{T,x}$. Therefore, $v_1, \ldots, v_{d-1}, u$ generate $m_{X,x}$ which proves that $x \in X$ is smooth. Thus, every point of W is a smooth point of X. This is a contradiction. □

Theorem 6.8. *Let Y be an affine algebraic variety. Then, it has an affine normalisation.*

Proof. Let $K = k(Y)$ be the function field of Y and let S be the integral closure of the ring $R = k[Y]$ in K. In commutative algebra it is proved that S is a finitely generated R-algebra and so a finite R-module, in particular, S is a finitely generated k-algebra (see [4, II, 5.2, Thm 4] or [5, V, Thm 9]). So, for some n there is a surjective k-homomorphism $\alpha\colon k[t_1, \ldots, t_n] \to S$ which implies that there is a prime ideal I in $k[t_1, \ldots, t_n]$ such that S is isomorphic to $k[t_1, \ldots, t_n]/I$ as a k-algebra. Put $X = V(I) \subseteq \mathbb{A}^n_k$. Thus, S is the coordinate ring of X and this induces a regular map $\phi\colon X \to Y$ which is finite and birational by construction. Moreover, X is normal.

We prove that ϕ is surjective. Let $y \in Y$ and let I_y be its maximal ideal in R. Then, since S is integral over R, there is a maximal ideal of

S whose intersection with R is I_y. Such a maximal ideal corresponds to a point $x \in X$ and so it is nothing but I_x. Thus, $\phi(x) = y$. Therefore, X is a normalisation of Y which is itself affine. □

Exercise 6.9. *Prove that the normalisation of a quasi-projective algebraic variety is unique if it exists.*

Exercise 6.10. *Find an example of a variety which is normal but not smooth (Hint: look at $V(t_1^2 + t_2^2 + t_3^2)$).*

7. Divisors

Definition 7.1. *Let X be a quasi-projective algebraic variety of dimension $d > 0$. A prime divisor on X is a subvariety of dimension $d-1$. A divisor is as $D = \sum_{i=1}^{n} a_i D_i$ where $a_i \in \mathbb{Z}$ and D_i are distinct prime divisors on X. In other words, a divisor is an element of the free abelian group generated by prime divisors on X. We show the group by* $\operatorname{Div}(X)$. *A divisor D is effective if all $a_i \geq 0$ and we denote effectivity by $D \geq 0$.*

Let $K = k(X)$. A valuation of K/k is a function $\mu\colon K - \{0\} \to \mathbb{Z}$ satisfying the following properties:

(i) $\mu(a) = 0$ if $a \in k$,
(ii) $\mu(fg) = \mu(f) + \mu(g)$, and
(iii) $\mu(f + g) \geq \min\{\mu(f), \mu(g)\}$.

The subset $S = \{f \in K \mid \mu(f) \geq 0\} \cup \{0\}$ is a subring of K which is called the valuation ring of μ, and the set $m = \{f \in K \mid \mu(f) > 0\} \cup \{0\}$ is the only maximal ideal of S.

Now let X be normal and D be a prime divisor on X. We define the valuation of D, μ_D as follows: since X is normal X is smooth on an open subset of D and the ideal of D is locally principal in some neighborhood. That is, there is an open affine subset U of X which intersects D and such that the ideal P of $D \cap U$ in $R = k[U]$ is principal. So, R_P is a normal ring of dimension one which is a DVR with a maximal ideal m.

Let $0 \neq f \in K$ and suppose that D is not contained in the non-regular points of f. So, $f \in R_P$ and define $\mu_D(f) = \max\{l \in \mathbb{N} \cup \{0\} \mid f \in m^l\}$. If D is contained in the non-regular points of f, then D is not contained in the non-regular points of $1/f$ and we define $\mu_D(f) = -\mu_D(1/f)$.

The definition of μ_D does not depend on U. Moreover, $\bigcap_{l=1}^{+\infty} m^l = 0$ by Nakayama's lemma, so $\mu_D(f)$ is well-defined. Finally, R_P is the valuation ring of μ_D and m coincides with the maximal ideal given by the valuation.

Definition 7.2. *Let X be a normal quasi-projective algebraic variety of dimension $d > 0$, and $f \in K = k(X)$. The divisor of f is defined as* $\operatorname{Div}(f) = \sum_{D_i} \mu_{D_i}(f) D_i$ *where D_i runs through the set of prime divisors of X. The sum is obviously finite.*

Definition 7.3. *Let X be a normal quasi-projective algebraic variety of dimension $d > 0$ and D, D' two divisors on X. We say that D is linearly equivalent to D' if $D - D' = \operatorname{Div}(f)$ for some $f \in K$. We show* *this equivalence by $D \sim D'$. Finally, we define the divisor class group of X as* $\mathrm{Cl}(X) = \mathrm{Div}(X)/\sim$.

Example 7.4. *Let $X = \mathbb{A}^n_k$ and D a prime divisor on X. The ideal I_D is a prime ideal in $k[t_1, \dots, t_n]$. Suppose that $D \subsetneq V(g)$. Then, $g \in I_D$ because I_D is a prime ideal. This means that I_D is a principal ideal which in turn implies that $D \sim 0$. Therefore,* $\mathrm{Cl}(X) = 0$.

Example 7.5. *Let $X = \mathbb{P}^n_k$ and D a prime divisor on X. Then, $I_D = \langle F \rangle$ for some homogeneous polynomial F of degree l. Let L be the prime divisor defined by the linear polynomial $s_0 = 0$. Since F/s_0^l is a rational function on X, $D - lL \sim 0$. In other words, every divisor is a multiple of L, therefore* $\mathrm{Cl}(X) = \mathbb{Z}$.

Exercise 7.6. *Let X be a normal quasi-projective algebraic variety and f a rational function on X. Prove that* $\mathrm{Div}(f) \geq 0$ *iff f is regular on X.*

Definition 7.7. *Let X be a topological space. A sheaf $\mathcal{F}$ on X is defined as follows:*

(i) for every open subset $U \subseteq X$, we have an abelian group $\mathcal{F}(U)$,
(ii) $\mathcal{F}(\emptyset) = 0$,
(iii) If $V \subseteq U$, we have a restriction homomorphism $r_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$,
(iv) r_{UU} is the identity,
(v) if $W \subseteq V \subseteq U$ are open subsets, then $r_{UW} = r_{VW} r_{UV}$,
(vi) if U is open and $U = \bigcup V_i$ is an open covering, and if $s \in \mathcal{F}(U)$ such that $r_{UV_i}(s) = 0$ for all i, then $s = 0$,
(vii) if U is open and $U = \bigcup V_i$ is an open covering, and if $s_i \in \mathcal{F}(V_i)$ such that $r_{V_i V_i \cap V_j}(s_i) = r_{V_j V_j \cap V_i}(s_j)$, then there is $s \in \mathcal{F}(U)$ such that $r_{UV_i}(s) = s_i$.

Elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ on U. Instead of $r_{UV}(s)$ it is often written $s|_V$.

Let X be a normal quasi-projective algebraic variety and D a divisor on X. Define

$$H^0(X, D) = \{f \in K \mid \mathrm{Div}(f) + D \geq 0\} \cup \{0\}$$

This is a vector space over k and it is known that it is finite dimensional if X is projective.

Exercise 7.8. *Let X be a normal quasi-projective algebraic variety and $D \sim D'$ linearly equivalent divisors on X. Prove that $H^0(X, D) \simeq H^0(X, D')$.*

Exercise 7.9. *Let X be a normal quasi-projective algebraic variety and D a divisor on X. Prove that the abelian groups $H^0(U, D|_U)$ define a sheaf on X. This is called the sheaf associated to D and is denoted by $\mathcal{O}_X(D)$.*

Exercise 7.10. *Let X be a normal quasi-projective algebraic variety and D a divisor on X. Prove that the rings $k[U]$ define a sheaf on X. This is called the structure sheaf of X and is denoted by $\mathcal{O}_X$. Note that this is the same as $\mathcal{O}_X(0)$.*

We now turn our attention to divisors on curves. For a divisor $D = \sum a_i D_i$ on a smooth projective curve X, define its degree as $\deg D = \sum a_i$. Here, a curve means a one dimensional algebraic variety.

Theorem 7.11. *Let X be a smooth projective curve and $D \sim D'$. Then, $\deg D = \deg D'$.*

We need to prove a couple of lemmas before we are able to prove this theorem.

Lemma 7.12. *Let $\phi\colon X \to Y$ be a non-constant regular map of smooth projective curves. Then, ϕ is a surjective finite map.*

Proof. Since X and Y are curves, $\phi(X)$ is dense in Y. Let $K = k(X)$, $L = k(Y)$, W an affine open subset of Y, $U = \phi^{-1}W$, and $R = k[W]$. Since $\dim X = \dim Y = 1$, K is a finite extension of L. In commutative algebra, it is proved that the integral closure of R in K, say S is a finite R-module. Let V be the affine variety defined by S.

For any open affine $U' \subseteq U$, R is inside $k[U']$ and since U' is smooth, $k[U']$ is normal. Therefore, S is inside $k[U']$ because $k[U']$ is integrally closed in K. This gives a regular map $U \to V$ which is birational because the function field of V is K. So, we get a rational map $\psi\colon V \dashrightarrow X$. It actually, is a regular map because V is smooth and X is projective. Now since $V \to W$ is finite and surjective, so ϕ is also surjective and $U = V = \phi^{-1}W$. □

Exercise 7.13. *Let X be a smooth projective curve. Prove that there is a finite regular map $\phi\colon X \to \mathbb{P}^1_k$.*

For a finite regular map $\phi\colon X \to Y$ of smooth quasi-projective curves and a divisor D on Y, we define the pullback of D as follows. Let $y \in Y$, $y \in W$ an affine neighborhood, $u \in k[W]$ a local parameter at $y \in Y$ such that $I_{\{y\}} = \langle u \rangle$ in $k[W]$, and $v = \phi^*(u)$ the regular function on $U = \phi^{-1}W$. Define $\phi^*(y) = \sum_{x \in U} \mu_x(v)x$. This is independent of the choice of u and W. Moreover, if $D = \sum_i a_i D_i$, put $\phi^* D = \sum_i a_i \phi^* D_i$. We also define the degree of ϕ to be $\deg(\phi) = [k(X) : k(Y)]$.

Lemma 7.14. *Let $\phi\colon X \to Y$ be a finite regular map of smooth projective curves and D a divisor on Y. Then,* $\deg \phi^* D = \deg(\phi) \deg D$.

Proof. Obviously, it is enough to prove the case $D = y$ where $y \in Y$ is a point of Y. Moreover, by taking an affine neighborhood of y, we may assume that X, Y are affine. Assume that $f^{-1}\{y\} = \{x_1, \dots, x_n\}$, $R = k[Y]$, $S = k[X]$, $L = k(Y)$, and $K = k(X)$. Moreover, let $I, I_1, \dots, I_n$ be the maximal ideals of $y, x_1, \dots, x_n$ in R and S, respectively.

Let $T = S_I$ be the localisation of S with respect to I. Here I may not be an ideal of S but $R - I$ is a multiplicative system of S and we can define localisation. Then, we have natural inclusions, $R \hookrightarrow S \hookrightarrow T$ and $R \hookrightarrow \mathcal{O}_{Y,y} \hookrightarrow T \hookrightarrow \mathcal{O}_{X,x_i}$. Now let $u \in R$ be a local parameter at $y \in Y$ with $I = \langle u \rangle$, and $J_i = T \cap u\mathcal{O}_{X,x_i}$. Thus, we the natural isomorphism

$$\frac{T}{\bigcap J_i} \simeq \frac{T}{J_1} \oplus \cdots \oplus \frac{T}{J_n}$$

because $J_i + J_j = T$ if $i \neq j$. The latter follows from the fact that $u\mathcal{O}_{X,x_i} = m_i^{\mu_{x_i}(u)}$ and $T \cap m_i$ is a maximal ideal of T where m_i is the maximal ideal of $\mathcal{O}_{X,x_i}$.

On the other hand,

$$\frac{T}{J_i} = \frac{T}{T \cap u\mathcal{O}_{X,x_i}} \simeq \frac{\mathcal{O}_{X,x_i}}{u\mathcal{O}_{X,x_i}}$$

because $\mathcal{O}_{X,x_i}$ is the localisation of T at I_i and $\frac{\mathcal{O}_{X,x_i}}{u\mathcal{O}_{X,x_i}}$ is a k-vector space with the basis $1, v_i, \dots, v_i^{\mu_{x_i}(u)-1}$ where $v_i \in T$ is a local parameter at x_i. Therefore,

$$\frac{T}{uT} \simeq \frac{\mathcal{O}_{X,x_1}}{u\mathcal{O}_{X,x_1}} \oplus \cdots \oplus \frac{\mathcal{O}_{X,x_n}}{u\mathcal{O}_{X,x_n}}$$

because $uT = \bigcap J_i$. In fact if $(g_1/h_1)u = \cdots = (g_n/h_n)u \in T$ where $g_i/h_i \in \mathcal{O}_{X,x_i}$, then it is easy to prove that $g_1/h_1 = \cdots = g_n/h_n \in T$ using the fact that $\langle h_1, \dots, h_n \rangle = T$. So, we have proved that

$$\dim_k \frac{T}{uT} = \sum \dim_k \frac{\mathcal{O}_{X,x_1}}{u\mathcal{O}_{X,x_1}} = \sum \mu_{x_i}(u) = \deg f^* y$$

To finish the proof, we should prove that $\deg(\phi) = \dim_k T/uT$. Let $\alpha_1, \dots, \alpha_d$ be a basis of K over L where $d = \deg(\phi)$. By multiplying each α with some power of u, we can assume that $\alpha_i \in T$. In particular, the α_i are linearly independent over $\mathcal{O}_{Y,y}$. Now since $\mathcal{O}_{Y,y}$ is a principal ideal domain and since T is an integral domain, T is a free $\mathcal{O}_{Y,y}$-module. In fact, $\alpha_1, \dots, \alpha_d$ form a basis for T over $\mathcal{O}_{Y,y}$ because there cannot be another element of T which is linearly independent of $\alpha_1, \ldots, \alpha_d$. Therefore, $\dim_k T/uT = \deg(\phi)$ and we are done. □

Proof. (of Theorem 7.11) Suppose that $D - D' = \mathrm{Div}(f)$ for some $f \in K = k(X)$. Define a rational map $\phi\colon X \to \mathbb{P}^1_k$ by $\phi = (1 : f)$. Since X is a smooth curve, ϕ is a regular map. By Lemma 7.12, ϕ is a surjective finite map. Let D'' be the divisor $(1 : 0) - (0 : 1)$ which is $\mathrm{Div}(s_1/s_0)$. Then, $\mathrm{Div}(f) = \phi^*(D'')$ because $f = \phi^*(s_1/s_0)$. By Lemma 7.14, $\deg \mathrm{Div}(f) = 0$ because $\deg D'' = 0$. □

So, for any smooth projective curve X we get a well-defined surjective group homomorphism $\deg\colon \mathrm{Cl}(X) \to \mathbb{Z}$. The kernel of this is shown by $\mathrm{Cl}^0(X)$.

Example 7.15. *Let X be a smooth projective curve and assume that there is a rational function f on X such that $\mathrm{Div}(f) = D_1 - D_2$ where D_1, D_2 are prime divisors as usual. We prove that $X \simeq \mathbb{P}^1_k$. Let $\phi\colon X \to \mathbb{P}^1_k$ be the rational map defined by $\phi = (1 : f)$ which is regular because X is smooth. On the other hand, $\phi^*(1 : 0) = D_1$. Thus, $\deg(\phi) = 1$ which means that $k(X) = k(\mathbb{P}^1_k)$. So, X and $\mathbb{P}^1_k$ are birational and therefore isomorphic.*

Theorem 7.16. *Let X be a smooth projective curve. Then, $k[X] = k$.*

Proof. Let $f \in k[X]$ and let $\phi\colon X \to \mathbb{P}^1_k$ be the regular function defined by $\phi = (1 : f)$. By Lemma 7.12, ϕ is constant or a surjective finite map. Since f is regular, $(0 : 1)$ is not in the image of ϕ. Therefore, ϕ and so f is a constant. □

Note that if we accept that every projective curve has a projective normalisation, then we can easily extend the previous theorem to any projective algebraic variety. In fact, if X is a projective algebraic variety and $f \in k[X]$, then f restricted to any projective curve $C \subseteq X$ gives a regular function $f|_C \in k[C]$. Now let $\pi\colon C' \to C$ be a projective normalisation of C. Then, $f\pi$ is a regular function on C' which should be constant. This is possible only if f is a constant.

We now prove a weak version of the Riemann-Roch theorem for curves.

Theorem 7.17. *Let X be a smooth projective curve and D a divisor on X. Then, $H^0(X, D)$ is a finite dimensional vector space over k. More precisely, $\dim_k H^0(X, D) \leq \deg D + 1$.*

Proof. If $\deg D < 0$, then $H^0(X, D) = 0$ and the theorem is proved in this case. So, we may assume that $\deg D \geq 0$. If $H^0(X, D) = 0$, then again the theorem is proved so we assume otherwise. In particular, we can replace D by one which is effective, that is, $D \geq 0$ because linearly equivalent divisors have the same degree by the theorems above.

We use induction on $\deg D$. If $\deg D = 0$, then $D = 0$ and so $H^0(X, D) = k$ by Theorem 7.16. Now assume that $\deg D > 0$ and $D = aD_1 + D'$ where $D' \geq 0$ does not contain D_1 and $a > 0$. Obviously, $H^0(X, (a-1)D_1 + D') \subseteq H^0(X, D)$.

Let u be a local parameter at D_1. Then, for any $f \in H^0(X, D)$, $u^a f$ is regular at D_1. So, we can define a linear map $L \colon H^0(X, D) \to k$ by putting $L(f) = (u^a f)(D_1)$. If L is constant, it means that

$$\mathrm{Div}(f) + (a-1)D_1 + D' \geq 0$$

if $f \in H^0(X, D)$ which means that $H^0(X, D) = H^0(X, (a-1)D_1 + D')$. We are done in this case. So, assume that L is not constant in which case it is surjective. Moreover, $H^0(X, (a-1)D_1 + D')$ is the kernel of L. So, $\dim_k H^0(X, D) \leq \deg D + 1$ by induction. □

8. Differential forms

Let X be an affine algebraic set. For each $f \in k[X]$ consider a formal symbol $\mathrm{d}\, f$. Let $\Omega[X]$ be the free $k[X]$-module generated by all the symbols $\mathrm{d}\, f$ modulo the following relations

$$\mathrm{d}(f+g) = \mathrm{d}\, f + \mathrm{d}\, g,$$
$$\mathrm{d}\, fg = f\, \mathrm{d}\, g + g\, \mathrm{d}\, f,$$
$$\mathrm{d}\, a = 0 \text{ for any } a \in k.$$

Elements of $\Omega[X]$ are called regular differential forms on X. So, an element $\omega \in \Omega[X]$ can be written in the form $\omega = \sum g_i\, \mathrm{d}\, f_i$ where $f_i, g_i \in k[X]$.[8.1]

If $U \subseteq X$ is an open affine subset, then there is a natural homomorphism $p\colon k[X] \to k[U]$ which induces a natural homomorphism $P\colon \Omega[X] \to \Omega[U]$. If $\omega \in \Omega[X]$ we usually write $\omega|_U$ instead of $P(\omega)$.

For every $f \in k[X]$ and $x \in X$, we have defined $d_x f \in T^*_{X,x}$ before. By putting $\mathcal{T} = \dot{\bigcup}_{x \in X} T^*_{X,x}$, to each $\omega = \sum g_i\, \mathrm{d}\, f_i \in \Omega[X]$ we can assign a function $X \to \mathcal{T}$ via $\omega(x) = \sum g_i(x)\, \mathrm{d}_x\, f_i$. However, if X is singular it may happen that $\omega \neq 0$ but the assigned function is zero.

Definition 8.1. *Let X be a quasi-projective algebraic set. A regular differential form is an object which locally on open affine subsets is a regular differential form as defined above. More precisely, a regular differential form ω on X is made of $\omega_1, \ldots, \omega_m$ where ω_i is a regular differential form on an open affine subset U_i, all the U_i cover X, and $\omega_i|_V = \omega_j|_V$ for any open affine subset $V \subseteq U_i \cap U_j$. The set of regular differential forms on X is denoted by $\Omega[X]$.*

One can check that if X is affine, then this definition coincides with the earlier definition.

Exercise 8.2. *Let X be a quasi-projective algebraic set. Prove that the abelian groups $\Omega[U]$ define a sheaf on X which is denoted by Ω. This sheaf is called the sheaf of regular differential forms.*

Example 8.3. *Let $X = \mathbb{A}^n_k$. For $f \in k[X]$, we have $\mathrm{d}\, f = \sum_{j=1}^n \frac{\partial f}{\partial t_j}\, \mathrm{d}\, t_j$. Then, $\Omega[X]$ is the free $k[X]$-module with the basis $dt_1, \ldots, dt_n$.*

Example 8.4. *Let $X = \mathbb{P}^1_k$. A regular differential form ω on X is determined by its restriction to the two affine open subsets $U = \mathbb{P}^1_k -$*

[8.1] There are other ways of defining differential forms which do not coincide with ours in the singular case but they are the same in the smooth case. See for example [4].

$\{(0:1)\}$ and $V = \mathbb{P}^1_k - \{(1:0)\}$. On U we take the variable t and on V the variable s. On $U \cap V$, we have $ts = 1$. Let $\omega|_U = f \,\mathrm{d}\, t$ and $\omega|_V = g \,\mathrm{d}\, s$. On $U \cap V$, $\mathrm{d}\, s = -\frac{1}{t^2} \,\mathrm{d}\, t$. So, on this intersection, $f \,\mathrm{d}\, t = -\frac{g}{t^2} \,\mathrm{d}\, t$. Therefore, $f(t) = -\frac{g(1/t)}{t^2}$ which implies that $f = g = 0$. So, $\Omega[X] = 0$.

Theorem 8.5. *Let X be a quasi-projective algebraic variety. Then, X is smooth iff every point $x \in X$ has an affine neighborhood U such that $\Omega[U]$ is a free $k[U]$-module.*

Proof. See [2] for an advanced proof. □

Let X be a smooth quasi-projective algebraic variety. According to Theorem 8.5, each point of X has an affine neighborhood U where $\Omega[U]$ is a free $k[U]$-module. More precisely, there are local parameter $u_1, \dots, u_d$ at some point of U such that $\mathrm{d}\, u_1, \dots, \mathrm{d}\, u_d$ form a basis of $\Omega[U]$ as a $k[U]$-module. Now for any $p \in \mathbb{N}$, we can take the exterior power $\bigwedge^p \Omega[U]$ which is a free $k[U]$-module whose basis consists of elements like $\mathrm{d}\, u_{i_1} \wedge \dots \wedge \mathrm{d}\, u_{i_p}$ where $u_{i_1}, \dots, u_{i_p}$ is a subset of $\{u_1, \dots, u_d\}$ with p elements and we ignore the order of the $i_1, \dots, i_p$. So, the basis has $\binom{d}{p}$ elements. Obviously, when $p > d$ we have $\bigwedge^p \Omega[U] = 0$.

Elements of $\bigwedge^p \Omega[U]$ are called regular differential p-forms on U. In the spirit of Definition 8.1, we can cover X with open affine subsets like U, take regular differential p-forms on the U and glue them together to get regular differential p-forms on X.

Definition 8.6. *Let X be a quasi-projective algebraic variety. A rational differential form ω is a regular differential form ω_U on some non-empty open subset U up to the following relation: we say that ω_U on U and $\omega'_{U'}$ on U' define the same rational differential form if their restriction to $U \cap U'$ is equal.*

Let U be an open affine subset of a quasi-projective algebraic variety X such that $\Omega[U]$ is a free $k[U]$-module with the basis $\mathrm{d}\, u_1, \dots, \mathrm{d}\, u_d$. Any rational differential form ω on X is determined by a regular differential form on some open subset of U. So, ω can be written as $\sum f_i \,\mathrm{d}\, u_i$ where f_i are rational functions.

Let ω be a rational differential form on a smooth quasi-projective curve X. Cover X by finitely many affine open subsets U_i such that ω is equal to $f_i \,\mathrm{d}\, u_i$ on U_i where $u_i \in k[U_i]$ is a local parameter at some point of U_i. On $U_i \cap U_j$, we have $f_i \,\mathrm{d}\, u_i = f_j \,\mathrm{d}\, u_j$ which implies that $\frac{f_i}{f_j} \,\mathrm{d}\, u_i = du_j$ and $\mathrm{d}\, u_i = \frac{f_j}{f_i} \,\mathrm{d}\, u_j$ hold on $U_i \cap U_j$. Since u_i and u_i are regular on $U_i \cap U_j$, on this set $\mathrm{Div}(\frac{f_i}{f_j}) = 0$, that is, $\mathrm{Div}(f_i) = \mathrm{Div}(f_j)$

on $U_i \cap U_j$. Now define the divisor $\mathrm{Div}(\omega)$ to be the divisor such that on U_i it is equal to $\mathrm{Div}(f_i)$. Obviously, $\mathrm{Div}(f\omega) = \mathrm{Div}(f) + \mathrm{Div}(\omega)$. So, no matter what rational differential form we choose we end up with the same divisor up to linear equivalence.

Definition 8.7. *Let X be a smooth quasi-projective curve. The canonical divisor K_X is defined up to linear equivalence as the divisor of a non-zero rational differential form.*

Definition 8.8. *Let X be a smooth projective curve. We define the genus of X by $g(X) = \dim_k H^0(X, K_X)$.*

Genus is the most important invariant in the classification of curves.

Example 8.9. *Let $X = \mathbb{P}^1_k$ and let $U = \mathbb{P}^1_k - \{(0:1)\}$ and $V = \mathbb{P}^1_k - \{(1:0)\}$. On U we take the variable t and on V the variable s. On $U \cap V$, we have $ts = 1$. Let ω be the rational differential form defined by $\mathrm{d}\, s$ on V. So, $\omega = \mathrm{d}\, s = -\frac{1}{t^2}\,\mathrm{d}\, t$. Therefore,* $\mathrm{Div}(\omega) = -2(1:0)$ *which means that $K_X = -2(1:0)$.*

Since $\deg K_X = -2$, *$H^0(X, K_X) = 0$ hence $g(X) = 0$.*

Exercise 8.10. *Let X and X' be two isomorphic smooth projective curves. Prove that $g(X) = g(X')$.*

9. Riemann-Roch theorem for curves

The Riemann-Roch problem is about calculating the dimension $\dim_k H^0(X, D)$ where X is a smooth projective variety and D a divisor on X. In general there is no easy formula for this. However, in low dimensions, eg, $1, 2, 3$, there is an explicit and fairly easy formula.

The following theorem for curves is the backbone of the theory of curves and its generalisation to higher dimension also plays a central role in algebraic geometry.

Theorem 9.1 (Riemann-Roch)**.** *Let D be a divisor on a smooth projective curve X. Then,*

$$\dim_k H^0(X, D) - \dim_k H^0(X, K_X - D) = \deg D + 1 - g(X)$$

The most conceptual and "easy" way to prove this theorem is via cohomology. In the section on cohomology we give the proof assuming certain theorems on cohomology such as the Serre duality. In Theorem 7.17, we proved a weak version of this theorem.

Example 9.2. *Let X be a smooth projective curve. Applying the Riemann-Roch theorem to $D = K_X$, we get*

$$\dim_k H^0(X, K_X) - \dim_k H^0(X, 0) = \deg K_X + 1 - g(X)$$

which implies that $\deg K_X = 2g(X) - 2$.

Example 9.3. *Earlier, we proved that $g(\mathbb{P}^1_k) = 0$. Let $D \geq 0$ be a divisor on X. The Riemann-Roch theorem implies that*

$$\dim_k H^0(X, D) = \deg D + 1$$

Example 9.4. *We prove that $\mathbb{P}^1_k$ is the only smooth projective curve with genus zero. Let X be a smooth projective curve with $g(X) = 0$. Let D be a prime divisor on X, that is, just a point. By the Riemann-Roch theorem,*

$$\dim_k H^0(X, D) = 2$$

which implies that there is a non-constant $f \in k(X)$ such that $D' :=$ $\operatorname{Div}(f) + D \geq 0$. *Thus,* $\operatorname{Div}(f) = D' - D$ *and since* $\deg \operatorname{Div}(f) = 0$, *$D'$ is also a prime divisor. Now by Example 7.15, $X \simeq \mathbb{P}^1_k$.*

Example 9.5. *An elliptic curve is a smooth projective curve X of $g(X) = 1$. So,* $\deg K_X = 0$. *Let $D \geq 0$ be an effective divisor on X. By the Riemann-Roch theorem*

$$\dim_k H^0(X, D) - \dim_k H^0(X, K_X - D) = \deg D$$

which says that if $D \neq 0$, then $\dim_k H^0(X, D) = \deg D$ *and if $D = 0$, then obviously* $\dim_k H^0(X, D) = 1$.

We can also calculate the canonical divisor K_X. By definition, $g(X) = \dim_k H^0(X, K_X) = 1$ and as mentioned $\deg K_X = 0$. So, there is an effective divisor D such that $D \sim K_X$ and $\deg D = 0$. Therefore, $D = 0$ and $K_X \sim 0$.

Example 9.6. *Let X be an elliptic curve and let $x \in X$. Let $\mathrm{Cl}^0(X)$ be the subgroup of $\mathrm{Cl}(X)$ consisting of divisors with degree zero. Now define a map $\theta\colon X \to \mathrm{Cl}^0(X)$ by $\theta(y) = y - x$ for any $y \in X$. We prove that this is one-to-one.*

The map θ is injective, otherwise $y - y'$ is the divisor of some rational function for some $y \neq y'$. But this is not possible by Example 7.15. Now let D be a divisor on X such that $\deg D = 0$. By the Riemann-Roch theorem

$$\dim_k H^0(X, D + x) = \deg(D + x) = 1$$

which implies that there is $f \in k(X)$ such that $D' := \mathrm{Div}(f) + D + x \geq 0$. So, $\deg D' = 1$. This means that D' is a single point hence $D \sim D' - x$, that is, $D = \theta(D')$. Thus, θ is bijective.

On the other hand, $\mathrm{Cl}^0(X)$ is an abelian group, so this makes X into an abelian group. This is the famous group law on elliptic curves.

Example 9.7. *For any smooth projective curve $X \subseteq \mathbb{P}^2_k$ of degree n, that is, defined by a homogeneous polynomial of degree n, it is possible to calculate its genus directly. In this case, $g(X) = (n-1)(n-2)/2$. This follows from the adjunction formula. In particular, we see that curves of degree 1 and 2 are all of genus zero and so they are isomorphic to the projective line by Example 9.4.*

If $n = 3$, then $g(X) = 1$ and so X is an elliptic curve according to Example 9.5. Another thing we can get form this is that two curves of different degree cannot be isomorphic except when one is of degree 1 and the other of degree 2.

10. Sheaves

See Definition 7.7 for the definition of a sheaf on a topological space.

Examples. Let X be a quasi-projective algebraic set.
(i) For any open subset U let $\mathcal{O}_X(U) = k[U]$. This defines a sheaf on X which is called the structure sheaf of X and is denoted by $\mathcal{O}_X$.
(ii) Let $\Omega(U) = \Omega[U]$. This also defines a sheaf Ω which is called the sheaf of regular differential forms.
(iii) When X is normal and D a divisor on X, $\mathcal{O}_X(D)(U) = H^0(U, D)$ defines a sheaf $\mathcal{O}_X(D)$ associated to the divisor D. In particular, the sheaf $\omega_X := \mathcal{O}_X(K_X)$ is called the canonical sheaf of X.
When X is smooth, $\mathcal{O}_X(D)$ is also called a line bundle.

Stalks. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a topological space X and $x \in X$. Define the stalk $\mathcal{F}_x = \varinjlim_{x\in U} \mathcal{F}(U)$ where U runs through the open subsets of X containing x.

Morphisms. A morphism $\rho\colon \mathcal{F} \to \mathcal{G}$ consists of homomorphisms $\rho(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ such that they commute with restriction maps of $\mathcal{F}$ and $\mathcal{G}$:

$$\begin{array}{ccc} \mathcal{F}(U) & \to & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \to & \mathcal{G}(V) \end{array}$$

where $V \subseteq U$. Any morphism $\rho\colon \mathcal{F} \to \mathcal{G}$ induces homomorphisms on the stalks: $\rho_x\colon \mathcal{F}_x \to \mathcal{G}_x$. The morphism ρ is called injective (resp. surjective, an isomorphism) if ρ_x is injective (resp. surjective, an isomorphism) for all $x \in X$.

Restriction to subsets. For an open subset $U \subseteq X$, we can restrict a sheaf $\mathcal{F}$ on X to U in an obvious way by putting $\mathcal{F}|_U(V) = \mathcal{F}(V)$ when $V \subseteq U$.

Gluing sheaves. If we cover X by open subsets U_i and if we have a sheaf $\mathcal{F}_i$ on each U_i, then by gluing these sheaves we can construct a sheaf on X subject to the following condition: for each i, j there is an isomorphism $\rho_{ij}\colon \mathcal{F}_i|_{U_i\cap U_j} \to \mathcal{F}_j|_{U_i\cap U_j}$ such that ρ_{ii} is the identity and $\rho_{il} = \rho_{jl}\rho_{ij}$ on $U_i \cap U_j \cap U_l$ for all i, j, l.

Sequences. A sequence of sheaves

$$\cdots \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}' \to \cdots$$

where each arrow is a morphism of sheaves is called exact if the sequence of stalks

$$\cdots \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to \cdots$$

is exact for all $x \in X$. In particular, an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is called a short exact sequence.

Direct image. Let $\phi \colon X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a sheaf on X. Then, we can define the direct image $\phi_*\mathcal{F}$ by putting $\phi_*\mathcal{F}(U) = \mathcal{F}(\phi^{-1}U)$. This is a sheaf on Y.

$\mathcal{O}_X$-modules. Let X be a quasi-projective algebraic set. A sheaf $\mathcal{F}$ on X is called an $\mathcal{O}_X$-module if for each open subset $U \subseteq X$, the group $\mathcal{F}(U)$ is a $k[U]$-module and the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the restriction homomorphism $k[U] \to k[V]$ for any open subset $V \subseteq U$.

A morphism $\rho \colon \mathcal{F} \to \mathcal{G}$ of sheaves on X is called a morphism of $\mathcal{O}_X$-modules if the two sheaves are $\mathcal{O}_X$-modules and for each open subset $U \subseteq X$, the homomorphism $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $k[U]$-modules.

Locally free sheaves. Let X be a quasi-projective algebraic set and $\mathcal{F}$ an $\mathcal{O}$-module. We say that $\mathcal{F}$ is locally free if each point of X has a neighborhood U such that $\mathcal{F}|_U \simeq \bigoplus \mathcal{O}_U$. If the sum is finite and r, the number of copies of $\mathcal{O}_U$, does not depend on U, then we call $\mathcal{F}$ a locally free sheaf of rank r.

Quasi-coherent sheaves. In algebraic geometry, the most important sheaves are the quasi-cohorent sheaves which are defined as follows. Let X be an affine algebraic set, $R = k[X]$ and M an R-module. Consider the disjoint union $\Theta = \dot{\bigcup}_{x \in X} M_x$ where M_x is the localisation of M with respect to the maximal ideal of x in R. We define $\tilde{M}(U)$ to be the set of functions $s \colon U \to \Theta$ such that for each $x \in U$, $s(x) \in M_x$, there is an open subset $V \subseteq U$ containing x, and elements $m \in M$ and $f \in R$ such that $s(y) = \frac{m}{f} \in M_y$ for every $y \in V$. The sheaf $\tilde{M}$ is the sheaf associated to M.

It is rather straightforward to prove that

(i) the stalk $(\tilde{M})_x \simeq M_x$,
(ii) for any $0 \neq f \in R$, $\tilde{M}(X_f) = M_f$ where $X_f = \{x \in X \mid f(x) \neq 0\}$ and M_f is the localisation of M with respect to the multiplicative system $\{1, f, f^2, \dots\}$,
(iii) $\tilde{M}(X) = M$. This follows from (ii) by putting $f = 1$.

Now let X be a quasi-projective algebraic set and $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is a quasi-coherent sheaf if every $x \in X$ has an open affine neighborhood U such that $\mathcal{F}|_U = \tilde{M}$ for some $k[U]$-module M. If for every x, the module M is finitely generated over $k[U]$, we say that $\mathcal{F}$ is coherent.

Examples. Let X be an affine algebraic set and $R = k[X]$. Then,

(i) $\tilde{R}$ is the structure sheaf $\mathcal{O}_X$,
(ii) $\widetilde{\Omega[X]}$ is the sheaf of regular differential forms on X,
(iii) if $I_Y \subseteq R$ is the ideal of a closed subset $Y \subseteq X$, $\tilde{I_Y}$ is called the ideal sheaf of Y.

Morphisms of quasi-coherent sheaves. Let X be an affine algebraic set and $R = k[X]$. If M, N are R-modules with an R-homomorphism $r\colon M \to N$, then we get a corresponding morphism of $\mathcal{O}_X$-modules $\rho\colon \tilde{M} \to \tilde{N}$. This gives an isomorphism between the category of R-modules and the category of quasi-coherent sheaves on X.

Tensor and sum of sheaves. Let U be an affine algebraic set, $R = k[U]$ and M, N two R-modules. Then, we define $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$ to be $(M \tilde{\otimes}_R N)$, and $\tilde{M} \oplus \tilde{N}$ to be $(M \tilde{\oplus} N)$. Now if X is a quasi-projective algebraic set and $\mathcal{F}$ and $\mathcal{G}$ two quasi-coherent sheaves on X, we define $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{F} \oplus \mathcal{G}$ to be the coherent sheaves on X such that locally on affine open subsets $U \subseteq X$ they are defined as above.

Tensor of sheaves of divisors. On a smooth quasi-projective algebraic set X, we have

$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D') \simeq \mathcal{O}_X(D + D')$$

11. Cohomology

Cohomology groups. Let X be a topological space, $\mathcal{F}$ a sheaf on X and $X = \bigcup_{i\in I} U_i$ a covering of X by open subsets U_i such that I is well-ordered and $U_i \neq U_j$ if $i \neq j$. We denote the covering by $\mathfrak{U}$. For any finite subset $\{i_0, \dots, i_p\} \subseteq I$, put $U_{i_0,\dots,i_p} = U_{i_0} \cap \dots \cap U_{i_p}$. Moreover, for each $p \in \mathbb{N} \cup \{0\}$ define

$$C^p = C^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0<\dots<i_p} \mathcal{F}(U_{i_0,\dots,i_p})$$

and a map $d_p \colon C^p \to C^{p+1}$ by sending an element $(s_{i_0<\dots<i_p}) \in C^p$ to an element $(t_{i_0<\dots<i_{p+1}}) \in C^{p+1}$ such that

$$t_{i_0<\dots<i_{p+1}} = \sum_{l=0}^{p+1} (-1)^l s_{i_0,\dots,\hat{i}_l,\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}}$$

One can check that $d_{p+1}d_p = 0$ and so we get a complex of abelian groups

$$0 \to C^0 \to C^1 \to C^2 \to \cdots$$

where $d_{-1} \colon 0 \to C^0$ is the trivial homomorphism. Now we define the (Čech) cohomology groups as

$$H^p(\mathfrak{U}, \mathcal{F}) = \ker d_p / \operatorname{im} d_{p-1}$$

Zero-th cohomology. Lets see what $H^0(\mathfrak{U}, \mathcal{F})$ is. By definition

$$C^0 = \prod_{i\in I} \mathcal{F}(U_i)$$

and the kernel of d_0 consists of those $(s_i) \in C^0$ such that

$$s_i|_{U_i \cap U_{i'}} = s_{i'}|_{U_i \cap U_{i'}}$$

which identifies an element of $\mathcal{F}(X)$ by the definition of a sheaf. Since the image of d_{-1} is 0 we get $H^0(\mathfrak{U}, \mathcal{F}) = \mathcal{F}(X)$.

Cohomology of quasi-coherent sheaves. From now on we assume that X is a quasi-projective algebraic set and $\mathcal{F}$ a quasi-coherent sheaf on X. Let $\mathfrak{U}$ be a covering of X by open affine subsets. We define the cohomology groups $H^p(X, \mathcal{F}) = H^p(\mathfrak{U}, \mathcal{F})$. It is proved that this does not depend on the choice of $\mathfrak{U}$.

Cohomology and dimension. Suppose that $d = \dim X$. If $p > d$, then $H^p(X, \mathcal{F}) = 0$.

Long exact sequence. For an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of quasi-coherent sheaves we have a long exact sequence of cohomology:

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to$$
$$H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{F}'') \to H^2(X, \mathcal{F}') \to \cdots$$

Cohomology of affine algebraic sets. Let X be an affine algebraic set. Then, $H^p(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $p > 0$. This is easy to see, just take the covering of X which consists of the one open subset X itself. Then, $C^0 = \mathcal{F}(X)$ and $C^1 = 0$.

Example. Let $X = \mathbb{P}^1_k$ and Ω be the sheaf of regular differential forms. Let $U = \mathbb{P}^1_k - \{(1 : 0\}$ and $V = \mathbb{P}^1_k - \{(0 : 1\}$. We know that $\Omega(U) = k[t]\,\mathrm{d}\,t$ and $\Omega(V) = k[s]\,\mathrm{d}\,s$ where t, s are the variables on U, V respectively satisfying $st = 1$ on $U \cap V$. Moreover, $\Omega(U \cap V) = k[t, \frac{1}{t}]\,\mathrm{d}\,t$. Now $C^0 = \Omega(U) \times \Omega(V)$ and $C^1 = \Omega(U \cap V)$. The map $d_0 \colon C^0 \to C^1$ is given by $d_0(f\,\mathrm{d}\,t, g\,\mathrm{d}\,s) = (f\,\mathrm{d}\,t - g\,\mathrm{d}\,s)|_{U \cap V}$ and its kernel is $H^0(X, \Omega) = 0$. The image of d_0 is the set of all the possible $(f\,\mathrm{d}\,t - g\,\mathrm{d}\,s)|_{U \cap V}$. Since $\mathrm{d}\,s = -\frac{1}{t^2}\,\mathrm{d}\,t$, we get elements of the form $f\,\mathrm{d}\,t + g(1/t)\frac{1}{t^2}\,\mathrm{d}\,t$. In particular, this contains all elements like $t^n\,\mathrm{d}\,t$ where $n \in \mathbb{Z}$ and $n \neq -1$. Therefore, $H^1(X, \Omega) = 1$.

Exercise. Let $X = \mathbb{P}^1_k$. Prove that $H^0(X, \mathcal{O}_X) = 1$ and $H^1(X, \mathcal{O}_X) = 0$.

Serre duality. Let X be a smooth projective variety of dimension d. Then, for any locally free sheaf $\mathcal{F}$ on X we have

$$H^p(X, \mathcal{F}) \simeq H^{d-p}(X, \check{\mathcal{F}} \otimes_{\mathcal{O}_X} \omega_X)$$

where $\check{\mathcal{F}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is the dual and $\omega_X = \mathcal{O}_X(K_X)$ is the canonical sheaf.

When $\mathcal{F} = \mathcal{O}_X(D)$ for a divisor D, then $\check{\mathcal{F}} = \mathcal{O}_X(-D)$ and so the Serre duality says that

$$H^p(X, \mathcal{O}_X(D)) \simeq H^{d-p}(X, \mathcal{O}_X(K_X - D))$$

Riemann-Roch theorem for curves. Here we show how to prove the Riemann-Roch theorem for curves using cohomology. Let X be a smooth projective curve and D a divisor on X. If $D = 0$, then

$$H^0(X, \mathcal{O}_X) - H^0(X, \mathcal{O}_X(K_X)) = 1 - g(X)$$

so we are done in this case. Now let $x \in X$ and assume that we have proved the theorem for D, we show that the theorem also holds for $D + x$. There is an obvious injective morphism $\mathcal{O}_X(-x) \to \mathcal{O}_X$. By taking the quotient, we get a short exact sequence

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to \mathcal{G} \to 0$$

for some sheaf $\mathcal{G}$. Tensoring the sequence with $\mathcal{O}_X(D + x)$ gives

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + x) \to \mathcal{F} \to 0$$

where $\mathcal{F} = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D + x)$. Now by the long exact sequence of cohomology we have

$$0 \to H^0(X, \mathcal{O}_X(D)) \to H^0(X, \mathcal{O}_X(D+x)) \to H^0(X, \mathcal{F}) \to$$

$$H^1(X, \mathcal{O}_X(D)) \to H^1(X, \mathcal{O}_X(D + x)) \to H^1(X, \mathcal{F}) \to 0$$

because $H^2(X, \mathcal{O}_X(D)) = 0$ since $\dim X = 1$. And linear algebra tells us that

$$\dim_k H^0(X, \mathcal{O}_X(D + x)) - \dim_k H^1(X, \mathcal{O}_X(D + x)) =$$

$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(X, \mathcal{O}_X(D)) + \dim_k H^0(X, \mathcal{F}) - \dim_k H^1(X, \mathcal{F})$$

One can easily check that $\mathcal{F}(U) = 0$ if $x \notin U$ and if you calculate the cohomology of $\mathcal{F}$ you see that $\dim_k H^0(X, \mathcal{F}) = 1$ and $H^1(X, \mathcal{F}) = 0$.

Therefore, putting all these together and using Serre duality we have

$$\dim_k H^0(X, \mathcal{O}_X(D+x)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D - x)) =$$

$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D)) + 1$$

and by induction

$$\dim_k H^0(X, \mathcal{O}_X(D+x)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D - x)) =$$

$$\deg D + 1 - g(X) + 1 = \deg(D + x) + 1 - g(X)$$

In the same way we can prove that if the theorem holds for $D + x$, then it also holds for D. This completes the proof.

References

[1] Atiyah, Macdonald; *Introduction to commutative algebra.*
[2] Hartshorne; *Algebraic geometry.*
[3] Matsumura; *Commutative ring theory.*
[4] Shafarevich; *Basic algebraic geometry I.*
[5] Zariski-Samuel; *Commutative algebra.* Volume 1.