Classification theory of algebraic varieties I

Introduction

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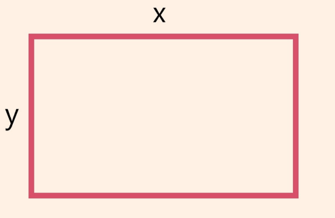
History: Babylon

Babylonian mathematicians solved equations thousands of years ago.

Suppose we have a **rectangle** with length t and width s.



Assume we know $t + s = a$ (sum of sides) and $ts = b$ (area).

Can we find t and s?

History: Babylon

From $s = \frac{b}{t}$ and $t + s = a$ we get

$$t + \frac{b}{t} = a$$

which gives the **quadratic equation**

$$t^2 - at + b = 0.$$

History: Galois theory

Later mathematicians tried to solve equations of higher degrees.

Let $k[t] =$ polynomials in variable t with coefficients in k.

Given $f \in k[t]$, we want to find its solutions.

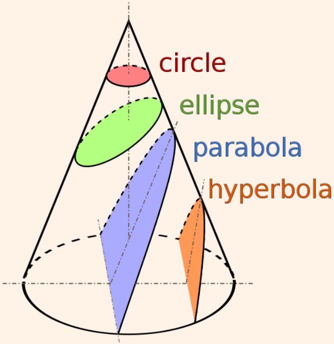
Example: taking $k = \mathbb{C}$, $t^2 + bt + c$ has solutions in $\mathbb{C}$: $(-b \pm \sqrt{b^2 - 4c})/2$.

Question: Can we find solutions of every f in terms of its coefficients (by radicals)?

Galois theory says: yes if $\deg f \leq 4$; not otherwise.

History: conic sections

Apollonius of Perga classified intersections of a plane with a cone, in the 3rd century BC.



Such conics are realised as curves defined by quadratic equations in the analytic geometry of Descartes.

History: Diophantine equations

A Diophantine equations a polynomial equation with coefficients in $\mathbb{Z}$.

Diophantus of Alexandria studied such equations in the 3rd Century.

Fermat studied such equations much later.

Example: $t_1^2 + t_2^2 + t_3^2 \in \mathbb{Z}[t_1, t_2]$ has only a trivial solution over $\mathbb{Z}$.

Example: $t_1^2 + t_2^2 - t_3^2 \in \mathbb{Z}[t_1, t_2]$ has many solutions over $\mathbb{Z}$, the Pythagorian triples.

Example: $t_1^m + t_2^m - t_3^m$, for $m \geq 3$, has only trivial solutions in $\mathbb{Z}$. This is Fermat's last theorem.

What is algebraic geometry?

Algebraic geometry is the study of solutions of systems of polynomial equations and associated geometric structures.

It is deeply related to many branches of mathematics but also to mathematical physics, computer science, etc.

It starts with choosing a **field** k.

For example, $k = \mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{F}_p$.

For simplicity we will take $k = \mathbb{C}$ from now unless otherwise stated.

Affine varieties: local geometry

Let $k[t_1, \ldots, t_n]$ = polynomials in variables t_i over k.

An (affine) **variety** is the common solutions of some $f_1, \ldots, f_r \in k[t_1, \ldots, t_n]$, that is

$$X = \{(a_1, \ldots, a_n) \quad | \quad f_j(a_1, \ldots, a_n) = 0, \quad \forall j\}$$

where $a_i \in k$.

Everything about X is encoded in the ring

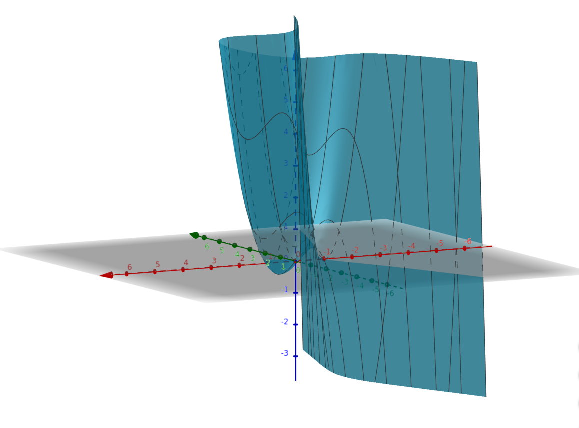
$$k[t_1, \ldots, t_n]/\langle f_1, \ldots, f_r \rangle.$$

Example: $t_2 - t_1^2$ defines a parabola in the plane $\mathbb{A}^2$.

Example: $t_1^2 + t_2^2 - 1$ defines a circle in $\mathbb{A}_k^2$.

Affine varieties: local geometry

Examples: the variety of $t_1^3 - t_1 + t_2^2 - t_3 = 0$ in $\mathbb{A}^3$:



Projective varieties: global geometry

The n-dimensional **projective space** is

$$\mathbb{P}^n_k = \{(a_0 : \cdots : a_n) \quad | \quad a_i \in k, \;\; a_j \neq 0 \text{ for some } j\}$$

subject to:

$$(a_0 : \cdots : a_n) = (ba_0 : \cdots : ba_n), \qquad\qquad \forall \;\; b \in k \setminus \{0\}.$$

$\mathbb{P}^n_k = \mathbb{A}^n_k \cup \mathbb{P}^{n-1}_k$ is a "compactification" of $\mathbb{A}^n_k$.

A **projective variety** in $\mathbb{P}^n_k$ is the solution set of some homogeneous polynomials

$$F_1, \ldots, F_r \in k[s_0, \ldots, s_n].$$

Projective varieties are locally affine varieties. This is similar to manifolds in differential geometry. The big difference: all manifolds look alike locally.

Goal: understand the local and global geometry of such X with respect to polynomial functions, i.e. classify them.

Maps

A variety X has a topology: its closed subsets are given by vanishing of polynomial equations.

Maps $X \dashrightarrow Y$ and **morphisms** $X \to Y$ between varieties are defined by rational functions.

$X \dashrightarrow Y$ is **birational** if it has an inverse map.

$X \to Y$ is an **isomorphism** if it has an inverse morphism.

Example: $\mathbb{P}^2 \to \mathbb{P}^2$ given by

$$(a_0 : a_1 : a_2) \mapsto (a_0^5 : a_1^5 : a_2^5).$$

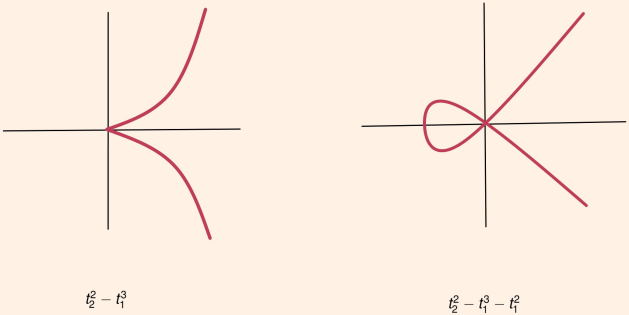
Example: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by

$$(a_0 : a_1 : a_2) \mapsto \left(\frac{1}{a_0} : \frac{1}{a_1} : \frac{1}{a_2}\right).$$

Singularities

Examples:

- $f = t_2^2 - t_1^3 \in \mathbb{C}[t_1, t_2]$ defines a **singularity** at $(0, 0)$.
- $f = t_2^2 - t_1^3 - t_1^2 \in \mathbb{C}[t_1, t_2]$ also defines a singularity at $(0, 0)$.



picture over real numbers

Singular points $x \in X$ are found by solving $\frac{\partial f}{\partial t_1}(x) = \frac{\partial f}{\partial t_2}(x) = 0$.

Singularities

A variety without singular points is called **smooth**.

Theorem (Hironaka, 1964)

For each variety Y there is a projective birational morphism $X \to Y$ from a smooth variety.

A smooth variety X can be viewed as a complex manifold.

Each smooth (or normal) variety X has a **canonical divisor** denoted $\mathbf{K_X}$ which is the divisor (zeros and poles) of a top degree rational differential form.

The sheaf $\mathcal{O}_X(K_X)$ associated to K_X is the canonical sheaf $\omega_X = \wedge^d \Omega_X$.

Example: $K_{\mathbb{P}^n} = (-n-1)H$ where H is a hyperplane.

Example: $X \subset \mathbb{P}^n$ a hypersurface of degree r, then

$$K_X = (r-n-1)H|_X.$$

Dimension one: curves

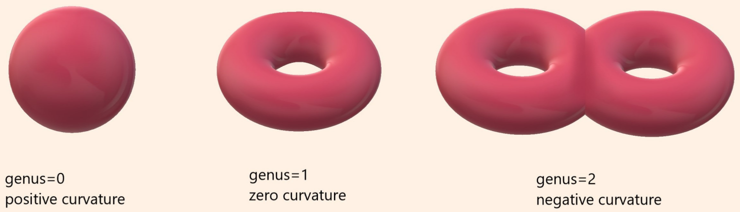
Assume X is a smooth projective variety of dimension one, i.e. a curve.

X is unique in its birational class.

The **genus** of X is

$$g = h^0(K_X) = h^1(\mathcal{O}_X) = \text{number of holes in } X$$

where X is considered as a compact Riemann surface.



Dimension one: curves

We have

$$\begin{cases} g = 0 & \Leftrightarrow \deg K_X < 0 & \Leftrightarrow X \text{ Fano} & \Leftrightarrow X \simeq \mathbb{P}^1 \\ g = 1 & \Leftrightarrow \deg K_X = 0 & \Leftrightarrow X \text{ Calabi-Yau} & \Leftrightarrow X \text{ elliptic} \\ g \geq 2 & \Leftrightarrow \deg K_X > 0 & \Leftrightarrow X \text{ canonically polarised} & \Leftrightarrow X \text{ general type} \end{cases}$$

We can study such X individually and also collectively in families.

Moduli (Riemann): curves of genus $g \geq 2$ are parametrised by points of a **moduli space** M_g of dimension $3g - 3$.

Moduli (Deligne-Mumford): M_g can be compactified by adding some curves with nodal singularities.

Curves have been extensively studied from the 19th century to this date.

Dimension two: surfaces

Let X be a smooth projective variety of dimension two, i.e. a surface.

Surfaces have a much more complicated geometry.

X is birational to infinitely many other surfaces. K_X may not be uniformly negative, trivial, or positive.

The idea is to transform X so that it has strong geometric properties.

There is a sequence of birational transformations

$$X = X_1 \to X_2 \to \cdots \to X_t =: Y$$

such that either

- K_Y is negative along fibres of some fibration $Y \to Z$, or
- K_Y is trivial along fibres of some fibration $Y \to Z$, or
- K_Y is positive.

Dimension two: surfaces

The above process is the classical **minimal model program** (MMP) for surfaces.

The next step is to study the outcomes Y in detail, e.g. form their moduli spaces.

The above classification scheme was developed in the 19th century and early 20th century by the Italian algebraic geometry school led by Castelnuovo and Enriques.

It is still ongoing research.

Dimension three

Fano and others studied varieties of dimension 3 in early 20th century.

It took many decades until a clear picture in dimension 3 emerged. In the 70's there was a wave of developments.

Iskovskikh-Manin developed techniques of Noether, Fano, Segre, giving counter-example to Lüroth problem by showing 3-fold quartics are non-rational.

Iitaka attempted to classify varieties according to their Kodaira dimension. He also studied open varieties.

Mori proved Hartshorne conjecture: $\mathbb{P}^n$ are the only smooth projective varieties with ample tangent bundle T_X.

Mori introduced bend and break technique and extremal rays.

By the early 90's the classification theory of 3-folds was well-developed.

Special varieties

Let X be a projective variety with "good" singularities.

$$\text{We say } X \text{ is } \begin{cases} \textbf{Fano} & \text{if } K_X \text{ is anti-ample, eg } \mathbb{P}^n \\ \textbf{Calabi-Yau} & \text{if } K_X \text{ is trivial, eg abelian varieties} \\ \textbf{canonically polarised} & \text{if } K_X \text{ is ample} \end{cases}$$

Example: $X \subset \mathbb{P}^n$ a hypersurface of degree r:

$$\begin{cases} r < n+1 & \Rightarrow X \text{ is Fano} \\ r = n+1 & \Rightarrow X \text{ is Calabi-Yau} \\ r > n+1 & \Rightarrow X \text{ is canonically polarised} \end{cases}$$

Example: X a smooth projective curve:

$$\begin{cases} genus = 0 \iff X \text{ Fano} & \iff X \simeq \mathbb{P}^1 \\ genus = 1 \iff X \text{ Calabi-Yau} & \iff X \text{ elliptic curve} \\ genus \geq 2 \iff X \text{ canonically polarised} & \iff X \text{ general type.} \end{cases}$$

Higher dimension

Let X be a smooth projective variety.

We like to run MMP giving a sequence of birational transformations

$$X = X_1 \overset{\text{div contraction}}{\dashrightarrow} X_2 \overset{\text{flip}}{\dashrightarrow} X_3 \dashrightarrow \cdots \dashrightarrow X_t = Y$$

This is a generalisation of the classical MMP for surfaces.

It is expected that either

- we have a Fano fibration $Y \to Z$, or
- we have a Calabi-Yau fibration $Y \to Z$, or
- Y is canonically polarised.

Running the MMP requires many local and global ingredients.

Higher dimension

Contractions: their existence was established by Kawamata and Shokurov.

Flips: Their existence was established by Mori in dimension 3, by Shokurov in dimensions 3,4, and by Hacon, M^{c}Kernan, Birkar, Cascini in any dimension.

Finite generation: existence of flips is equivalent to a local version of this fact:

$$\bigoplus_{m\geq 0} \mathcal{O}_X(mK_X)$$

is a finitely generated $\mathbb{C}$-algebra.

Historical summary:

- dimension 2: (Castelnuovo, Enriques)(Zariski, Kodaira, Shafarevich, etc) 1900,
- dimension 3 (Kawamata, Kollár, Miyaoka, Mori, Reid, Shokurov)(Fano, Hironaka, Iitaka, Iskovskikh, Manin, etc) 1970's-1990's,
- any dimension for X of general type (BCHM=B-Cascini-Hacon-M^{c}Kernan, after Shokurov, etc) 2006.

Boundedness and moduli

After the MMP the idea is to classify the outcomes Y, e.g. form their moduli spaces.

Hacon-McKernan-Xu: canonically polarised varieties Y of fixed dimension d and fixed volume K_Y^d form a bounded family.

Birkar: Fano varieties of fixed dimension and bounded singularities form a bounded family.

Birkar: Calabi-Yau varieties of fixed dimension polarised by ample divisor of fixed volume form a bounded family.

Kollár, Alexeev, etc developed a general theory of moduli of varieties.

Combining all these: moduli spaces exist for canonically polarised varieties and for polarised Calabi-Yau varieties.

Blum, Li, Liu, Xu, Zhuang and others have tried to construct moduli of K-stable Fano's after work of Yau, Tian, Donaldson, Chen, Sun and others.

Arithmetic geometry

For a variety X over a field k, let $X(k) = \{k\text{-rational points on } X\}$.

Esnault: X smooth Fano, k a finite field, then $X(k) \neq \emptyset$.

Graber-Harris-Starr: X rationally connected, k function field of a curve, $X(k) \neq \emptyset$.

Mordell-Weil: X abelian variety, k number field, $X(k)$ is a finitely generated abelian group.

Conjecture (Lang, Campana, etc):
If X is a smooth Fano or Calabi-Yau variety over a number field k, then there is a finite extension $k \subseteq k'$ such that $X(k')$ is dense.

Fibrations are even harder to understand.

Conjecture:
If $X \to \mathbb{P}^2$ is a conic bundle over a number field k, then there is a finite extension $k \subseteq k'$ such that $X(k')$ is dense.