# Classification theory of algebraic varieties III

# Generalised pairs

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Monroe H. Martin Lectures, Johns Hopkins University, February 2021

## Pairs

We work over $\mathbb{C}$.

We discussed pairs in the previous lecture.

Pairs and their singularities play a fundamental role in birational algebraic geometry.

A **pair** $(X, B)$ consists of a normal variety $X$ and a boundary divisor $B$ with coefficients in $[0, 1]$.

Singularities of $(X, B)$ are defined by taking a log resolution $\phi\colon W \to X$ and writing $K_W + B_W = \phi^*(K_X + B)$.

We say $(X, B)$ is **$\epsilon$-lc** if every coefficient of $B_W$ is $\leq 1 - \epsilon$.

Lc means 0-lc. Klt means $\epsilon$-lc for some $\epsilon > 0$.

## Why pairs?

### Open varieties:

In the 70's Iitaka studied smooth **open** varieties $X$.

He compactified $X$ to $Y$ so that $B := Y \setminus X$ has simple normal crossings.

The idea is that the geometry of $(Y, B)$ reflects the geometry of $X$.

Note $(Y, B)$ is not unique in dimension $\geq 2$.

This approach evolved into the theory of pairs.

### Adjunction:

Suppose $X$ is a smooth variety and $B$ is a smooth prime divisor on $X$.

The adjunction formula says $K_B = (K_X + B)|_B$.

## Why pairs?

Generalisations of this formula play a central role in birational geometry.

Often one derives non-trivial statements about $(X, B)$ from $B$ allowing proofs by induction on dimension.

Thus one is led to study pairs such as $(X, B)$.

Canonical bundle formula:

Suppose $X$ is a smooth projective variety and $f \colon X \to Z$ is a contraction.

Suppose $K_X \sim f^*L$ for some $\mathbb{Q}$-divisor $L$.

## Why pairs?

The **canonical bundle formula** says

$$K_X \sim f^*(K_Z + B + M)$$

where $B$ is the discriminant divisor and $M$ is the moduli divisor.

The classical example is Kodaira's canonical bundle formula when $X$ is a surface and $f$ is an elliptic fibration.

The canonical bundle formula allows one to investigate the geometry of $X$ from that of $(Z, B + M)$.

It is possible to choose $M$ so that we can consider $(Z, B + M)$ as a pair.

We will see that it is better to consider it as a generalised pair.

Quotient varieties:

Suppose $X$ is a smooth variety and $G$ is a finite group acting on $X$.

## Why pairs?

Let $Y = X/G$ and let $\pi\colon X \to Y$ be the quotient map.

By Hurwitz formula, $K_X = \pi^*(K_Y + B_Y)$ for some divisor $B_Y$.

It is then natural to study $(Y, B_Y)$ rather than just $Y$.

### Using pairs:

We illustrate the power of pairs by an example which frequently comes up in inductive statements.

Suppose $(X, B)$ is a projective Klt pair and $S$ is a normal prime divisor on $X$.

Suppose $L$ is a Cartier divisor such that

$$A := L - (K_X + B + S)$$

is ample.

## Why pairs?

Consider the exact sequence

$$H^0(X, L) \to H^0(S, L|_S) \to H^1(X, L - S).$$

Note $L - S = K_X + B + A$.

Since $A$ is ample and $(X, B)$ is Klt, $H^1(L - S) = 0$ by the Kawamata-Viehweg vanishing theorem.

So sections of $L|_S$ can be lifted to sections of $L$.

This is very useful when we want to show the linear system $|L|$ is non-empty or that it is base point free.

It is then natural to study $(X, B + S)$.

It is no surprise that the modern classification theory is centred around pairs.

## Generalised pairs

A **generalised pair** is roughly a pair together with a nef divisor on some birational model.

Polarised varieties:

Consider a projective variety $X$ and an ample divisor $M$ on it.

We say $X$ is polarised by $M$.

For example, $M$ can be a very ample divisor determining an embedding of $X$ into some projective space.

Polarised varieties play a central role in moduli theory, e.g. when $M = K_X$ is ample.

For applications it is important to allow more general polarisations when $M$ is only nef.

$M$ is not necessarily effective so $(X, M)$ is not a usual pair.

## Generalised pairs

Canonical bundle formula:

Suppose $(X, B)$ is a projective lc pair and $f \colon X \to Z$ is a contraction.

Suppose $K_X + B \sim_{\mathbb{Q}} f^*L$ for some $\mathbb{Q}$-divisor $L$.

Then the canonical bundle formula says

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z).$$

Usually $M_Z$ is not nef but it is nef on some birational model of $Z$.

So we consider $(Z, B_Z + M_Z)$ as a generalised pair.

Adjunction for lc centres:

Assume $(X, B)$ is a projective lc pair and $V$ is the normalisation of an lc centre.

## Generalised pairs

Then we can write
$$(K_X + B)|_V \sim_{\mathbb{Q}} K_V + C + N$$
where $C$ is a divisor with coefficients in $[0, 1]$ and $N$ is the pushdown of a nef divisor.

Then $(V, C + N)$ is a generalised pairs.

Varieties with nef $-K_X$:

Consider normal projective varieties $X$ with $-K_X$ nef.

They constitute an important class but are not readily treatable using pairs.

Letting $M := -K_X$ we get a generalised pairs $(X, M)$ with $K_X + M = 0$.

So we get a generalised log Calabi-Yau pair which can be treated using generalised pairs.

## Definition of generalised pairs

A projective **generalised pair** consists of

- a normal projective variety $X$,
- a $\mathbb{Q}$-divisor $B \geq 0$ on $X$, and
- a birational contraction $\phi\colon X' \to X$ and a nef $\mathbb{Q}$-divisor $M'$ on $X'$

such that $K_X + B + M$ is $\mathbb{Q}$-Cartier where $M := \phi_* M'$.

We specify $X', M'$ only up to birational transformations, i.e. we can replace $X'$ with a higher model and $M'$ with its pullback.

Assuming $\phi$ is a log resolution of $(X, B)$ we can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M).$$

We say $(X, B + M)$ is generalised lc (resp. generalised klt) (resp. generalised $\epsilon$-log canonical) if each coefficient of $B'$ is $\leq 1$ (resp. $< 1$)(resp. $\leq 1 - \epsilon$).

Remark: generalised pairs are also defined in the relative setting $X \to Z$ where $M'$ is nef over $Z$.

## Example

Let $X = \mathbb{P}^2$ and let $\phi\colon X' \to X$ be the blowup of a point $x$.

Let $E'$ be the exceptional divisor, $L$ a line passing through $x$ and $L'$ its birational transform.

If $B = 0$ and $M' = L'$, then we can calculate $B' = 0$ hence $(X, B + M)$ is generalised klt.

If $B = 0$ and $M' = 2L'$, then we can calculate $B' = E'$ hence $(X, B + M)$ is generalised lc.

If $B = 0$ and $M' = 3L'$, then $(X, B + M)$ is not generalised lc because in this case $B' = 2E'$.

If $B = L$ and $M' = 2L'$, then $(X, B + M)$ is not generalised lc because in this case $B' = L' + 2E'$.

## Applications: effective Iitaka fibrations

Let $W$ be a smooth projective variety of Kodaira dimension $\kappa(W) \geq 0$.

The Kodaira dimension $\kappa(W)$ is the largest number $\kappa \in \{-\infty, 0, 1, \ldots, \dim X\}$ such that

$$\limsup_{m \in \mathbb{N}} \frac{h^0(X, mK_W)}{m^\kappa} > 0.$$

By Iitaka, for sufficiently divisible $m \in \mathbb{N}$, the system $|mK_W|$ defines the **Iitaka fibration** $W \dashrightarrow X$.

The very general fibres $F$ of $W \dashrightarrow X$ have Kodaira dimension zero and $\dim X = \kappa(W)$.

Birkar-Zhang: we can choose $m$ bounded if certain invariants of $F$ are bounded.

A canonical bundle type formula gives divisors $B \geq 0$ and nef $M$ s.t. can assume

$$H^0(W, mK_W) \simeq H^0(X, m(K_X + B + M))$$

for $m$ divisible by some fixed number.

## Applications: effective Iitaka fibrations

So it is enough to find bounded $m$ s.t. $|m(K_X + B + M)|$ defines a birational map.

This follows from the next more general statement.

Birkar-Zhang: Let $d, r \in \mathbb{N}$ and let $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Assume

- $(X, B)$ is a projective lc pair of dimension $d$,
- the coefficients of $B$ are in $\Phi$,
- $rM$ is a nef Cartier divisor, and
- $K_X + B + M$ is big.

Then $|m(K_X + B + M)|$ defines a birational map for some bounded $m \in \mathbb{N}$.

For usual pairs, that is when $M = 0$, this was proved by Hacon-McKernan-Xu.

## Applications: boundedness of complements and of Fano varieties

Birkar: Let $d \in \mathbb{N}$ and $\Phi \subset [0,1]$ be a finite set of rational numbers. Assume

- $(X, B)$ is a projective lc pair of dimension $d$,
- the coefficients of $B$ are in $\Phi$,
- $X$ is of Fano type, and
- $-(K_X + B)$ is nef.

Then there is an $n$-complement of $K_X + B$ for some bounded $n$.

The proof reduces to when $K_X + B \sim_{\mathbb{Q}} 0$ along a fibration $f\colon X \to Z$.

Applying the canonical bundle formula we can write

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z).$$

We construct complements for $K_Z + B_Z + M_Z$ and pullback to $X$.

The above result is then used to prove the BAB conjecture and various other results.

## Applications: varieties with nef anti-canonical divisor

Birkar, Di Cerbo, Svaldi: Let $\epsilon \in \mathbb{R}^{>0}$ and consider projective varieties $X$ such that

- $(X, B)$ is $\epsilon$-lc of dimension 3 for some boundary $B$,
- $-(K_X + B)$ is nef, and
- $X$ is rationally connected.

Then such $X$ are bounded up to isomorphism in codimension one.

The rational connectedness assumption cannot be removed.

The idea: letting $M := -(K_X + B)$ we get an $\epsilon$-lc generalised pair $(X, B + M)$ with $K_X + B + M = 0$.

So $(X, B + M)$ is a generalised log Calabi-Yau pair.

This is preserved under running MMP, so can reduce to boundedness of Fano fibrations.

## Other applications

Moraga: termination of flips holds for pseudo-effective lc pairs of dimension 4.

Hacon-Moraga: termination of flips for generalised pairs with weak Zariski decompositions follows from termination in lower dimension for generalised klt pairs.

Han-Li: existence of weak Zariski decompositions is equivalent to existence of minimal models for generalised pairs.

Birkar and Filipazzi-Svaldi: for pair $(X, B)$ and contraction $X \to Z$ with $-(K_X + B)$ nef over Z, the non-klt locus $\mathrm{Nklt}(X, B)$ has at most two connected components near each fibre of $X \to Z$.

Birkar-Chen: for $\epsilon \in \mathbb{R}^{>0}$, if $X$ is $\epsilon$-lc and $X \to Z$ is a toric Fano fibration, then the fibres have bounded multiplicities.

Birkar: if $X$ is Calabi-Yau and $N$ is a big Weil divisor on $X$, then $|mN|$ defines a birational map for $m$ depending on $\dim X$.

Shokurov: proved more general results about boundedness of complements.

## Open problems

Conjecture: We can run MMP on any projective generalised lc pair terminating with a minimal model or a Mori fibre space.

In the klt case only the termination part is unknown but in the lc case the main steps of the MMP (contractions, flips) are also unknown.

Conjecture: (Lazić-Peternell) Assume $(X, B)$ is projective klt and $M$ is a nef divisor s.t. $K_X + B$ is pseudo-effective and $K_X + B + M$ is nef. Then $K_X + B + M \equiv L$ for some semi-ample divisor $L$.

Let $d, p \in \mathbb{N}$, $v \in \mathbb{Q}^{>0}$. Let $\mathfrak{G}(d, p)$ be the set of projective generalised lc pairs $(X, B + M)$ s.t.

- $\dim X = d$,
- $pB$ is integral, and
- $pM'$ is Cartier where $M'$ is the nef part of the pair.

Let $\mathfrak{F}(d, p, v)$ be the set of those $(X, B + M) \in \mathfrak{G}(d, p)$ s.t. $K_X + B + M$ is ample with $\mathrm{vol}(K_X + B + M) = v$.

## Open problems

Conjecture: the set

$$\{\mathrm{vol}(K_X + B + M) \mid (X, B + M) \in \mathfrak{G}(d, p)\}$$

satisfies DCC.

Conjecture: There is $m \in \mathbb{N}$ s.t. $m(K_X + B + M)$ is very ample for every $(X, B + M) \in \mathfrak{F}(d, p, v)$.

In particular, the $(X, B)$ belong to a bounded family.

The previous two conjectures were verified in dimension 2 by Filipazzi.

Conjecture: $d \in \mathbb{N}, \epsilon \in \mathbb{R}^{>0}$. Consider the set of projective generalised $\epsilon$-lc pairs $(X, B + M)$ s.t. $K_X + B + M \equiv 0$ and $X$ is rationally connected. Then such $X$ form a bounded family.

## Classification of generalised pairs

We already know a lot about the geometry of generalised pairs.

It is clear that regardless of applications, generalised pairs are of independent interest.

Goal: Classify projective generalised pairs $(X, B+M)$.

Example: Fix a smooth projective variety $X$ and consider all the Cartier divisors $M \sim_{\mathrm{alg}} 0$ up to linear equivalence. The generalised pairs $(X, M)$ are parametrised by $\mathrm{Pic}^0(X)$. Here we need to identify $(X, M_1)$ and $(X, M_2)$ if $M_1 \sim M_2$.

In general, we can focus on special classes, e.g. when $K_X + B + M$ is ample, numerically trivial, or anti-ample.

There is a lot to explore.