Classification theory of algebraic varieties II

Log Calabi-Yau fibrations

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Setup

We work over $\mathbb{C}$.

$\mathbb{A}^n$ is the usual n-dimensional affine space.

$\mathbb{P}^n$ is the n-dimensional projective space.

We work with quasi-projective varieties $X \subset \mathbb{P}^n$.

Algebraic geometry is largely concerned with classification of such varieties.

Fano varieties, Calabi-Yau varieties, and their generalisations play a fundamental role.

Such varieties fit into the framework of log Calabi-Yau fibrations $(X, B) \to Z$.

This framework covers an incredible range of notions, results, and open problems.

Fano varieties

A **Fano variety** is a projective variety X with good singularities and ample $-K_X$.

Examples:

- $\mathbb{P}^n$.
- Hypersurfaces $X \subset \mathbb{P}^n$ of degree $\leq n$.
- Conic sections $X = C \cap H \subset \mathbb{P}^3$, $\deg C = 2$ and $\deg H = 1$.
- Complete intersections $X = V_1 \cap \cdots \cap V_r \subset \mathbb{P}^n$, $\sum \deg V_i \leq n$.
- Ample model X of $-K_W$ for projective toric varieties W. Here $W \dashrightarrow X$.
- Weighted projective spaces $\mathbb{P}(a_0, \ldots, a_n)$.
- Complete intersections in $\mathbb{P}(a_0, \ldots, a_n)$ with appropriate degrees.
- Grassmannians $Gr(l, n)$.

Fano varieties

$\mathbb{P}^1$ is the only Fano variety of dimension 1.

Del Pezzo surfaces are Fano varieties of dimension 2.

Fano studied Fano varieties of dimension 3. Crucial in extending birational geometry to higher dimensions.

Iskovskikh-Manin developed techniques of Noether, Fano, Segre, giving counter-example to Lüroth problem by showing 3-fold quartics are non-rational.

Shokurov showed $|-K_X|$ has smooth members for smooth Fano 3-folds X. Origins of complements theory.

Mori proved Hartshorne conjecture: $\mathbb{P}^n$ are the only smooth projective varieties with ample T_X.

Mori introduced bend and break technique and extremal rays. This led to extremal rays and Mori theory or MMP.

Fano varieties

Shokurov introduced theory of complements: for X Fano, he studied nice elements in $|-nK_X|$. Later developed with Prokhorov.

Birkar proved Shokurov conjecture: $|-nK_X|$ has a nice element for n depending only on $\dim X$.

Birkar proved Borisov-Alexeev-Borisov conjecture: ϵ-lc Fano varieties of fixed dimension form a bounded family.

Shramov, Prokhorov + Birkar showed $\mathrm{Bir(X)}$ is Jordan for X Fano.

Blanc, Lamy, Zimmermann: $\mathrm{Bir}(\mathbb{P}^\mathrm{n})$ is not simple, $n \geq 3$. The case $n = 2$ was done earlier by Cantat, Lamy.

Fano fibrations

A **Fano fibration** is a "fibration" $X \to Z$ where X has good singularities and $-K_X$ is ample over Z.

Examples:

- X Fano, $Z = pt$.
- $X \to Z$ a $\mathbb{P}^n$ bundle.
- $X \to Z$ a Mori fibre space.
- $X \to Z$ a flipping contraction.
- $X \to Z$ a divisorial contraction.
- $X \to Z$ identity (singularity theory).

General question: how the geometry of X is related to the geometry of Z and fibres of $X \to Z$?

Fano fibrations

A flipping contraction is a birational Fano "fibration" $f\colon X \to Z$ not contracting any divisor.

They are important ingredients of the minimal model program.

The flip of f exists if and only if

$$\bigoplus_{m\geq 0} f_*\mathcal{O}_X(mK_X)$$

is a finitely generated $\mathcal{O}_Z$-algebra.

Mori showed flips exist in dimension 3 (for terminal singularities).

Shokurov showed flips exist in dimension ≤ 4.

Hacon, M^c^Kernan, Birkar, Cascini showed flips exist in all dimensions.

Calabi-Yau varieties

A **Calabi-Yau** variety is a projective variety X with good singularities and $K_X \equiv 0$.

Examples:

- Elliptic curves.
- Hypersurfaces $X \subset \mathbb{P}^n$ of degree $n+1$.
- K3 surfaces, abelian surfaces, Enriques surfaces, Kummer surfaces.
- Abelian varieties.
- Complete intersections $X = V_1 \cap \cdots \cap V_r \subset \mathbb{P}^n$, $\sum \deg V_i = n+1$.
- Complete intersections in $\mathbb{P}(a_0, \ldots, a_n)$ similarly defined.
- Smooth element X in $|-K_Y|$ for smooth Fano Y.

Calabi-Yau varieties

Mirror symmetry: smooth Calabi-Yau varieties come in pairs.

Mirror symmetry is a fundamental tool in string theory and enumerative geometry.

Conjecture: X a smooth Calabi-Yau variety, L a nef divisor, then $L \equiv M$ for some semi-ample divisor M.

A **Calabi-Yau fibration** is a "fibration" $X \to Z$ such that $K_X \equiv 0/Z$.

Examples:

- X Calabi-Yau variety, $Z = pt$.
- F Calabi-Yau variety, $X = F \times Z \to Z$ projection.
- Elliptic surfaces $X \to Z$.
- Hypersurfaces $X \subset \mathbb{P}^n \times Z$ of relative degree $n + 1$.

Calabi-Yau varieties

Given a Calabi-Yau fibration $X \to Z$, consider Calabi-Yau fibrations $X' \to Z$ isomorphic to X in codimension one:

- Kawamata: $X \dashrightarrow X'$ is decomposed into flops.
- Conjecture (Morrison-Kawamata): up to isomorphism, there are finitely many such $X' \to Z$.

The minimal model program (MMP)

W, a smooth projective variety.

Run the **MMP**: this is a sequence of birational transformations

$$W = W_1 \overset{\text{div contraction}}{\dashrightarrow} W_2 \overset{\text{flip}}{\dashrightarrow} W_3 \dashrightarrow \cdots \dashrightarrow W_r = X$$

designed to make the canonical divisor as positive as possible

Each step is a Fano fibration.

The MMP is known in $\dim \leq 3$ and in many cases in all dimensions.

Conjecture (Termination): The MMP end with some X after finitely many steps.

Conjecture (Abundance): Either

- X admits a Fano fibration $X \to Z$, or
- X admits a Calabi-Yau fibration $X \to Z$, or
- K_X is "nearly" ample.

Singularities

A **pair** (X, B) consists of a normal variety X and a boundary divisor B with coefficients in $[0, 1]$.

Singularities of (X, B) are defined by taking a log resolution $\phi\colon W \to X$ and writing $K_W + B_W = \phi^*(K_X + B)$.

We say (X, B) is ϵ**-lc** if every coefficient of B_W is $\leq 1 - \epsilon$.

Lc means 0-lc. Klt means ϵ-lc for some $\epsilon > 0$.

Examples:

- lc: X a smooth variety and B a simple normal crossing divisor.
- lc: X a smooth surface and B a nodal curve (but B cuspidal not lc).
- lc not klt: X cone over elliptic curve.
- $\frac{2}{n}$-lc: X cone over rational curve of degree n.

Log Calabi-Yau fibrations

A **log Calabi-Yau fibration** $(X, B) \to Z$ consists of an lc pair (X, B) and a "fibration" $X \to Z$, with $K_X + B \equiv 0$ over Z.

Examples:

- Calabi-Yau varieties and Calabi-Yau fibrations.
- Fano varieties and fibrations: $X \to Z$ Fano fibration, $B \equiv -K_X/Z$ general.
- Toric fibrations $(X, B) \to Z$.
- MMP on an lc pair (W, C) gives a log Calabi-Yau fibration $(X, B) \to Z$.
- Singularities: (X, B) lc pair, $X = Z$.

General questions on log Calabi-Yau fibrations $(X, B) \to Z$:

- Under what conditions are they bounded?
- How the linear systems $|-nK_X|/Z$ behave?

Log Calabi-Yau fibrations

- How singularities on X, Z behave?
- Under what conditions they admit moduli spaces?
- How do they look like topologically?
- Under what conditions is there a mirror symmetry?
- How their geometry is related to their arithmetic, over number fields?

Boundedness and moduli

Birkar: (d, r, ϵ)-Fano type log Calabi-Yau fibrations $(X, B) \to Z$ are bounded ($\epsilon > 0$).

Conjecture (M^{c}Kernan-Prokhorov): ϵ-lc rationally connected projective varieties X with $-K_X$ nef, are bounded ($\epsilon > 0$).

Conjecture (Yau): smooth strictly Calabi-Yau varieties X of fixed dimension > 2 are bounded. Strict means simply connected, $K_X \sim 0$, $h^i(\mathcal{O}_X) = 0$ for $0 < i < \dim X$.

Birkar, Di Cerbo, Svaldi: Yau conjecture holds for elliptic Calabi-Yau varieties with a rational section.

Satake, Mumford, Alexeev, etc: compact moduli spaces exist for polarised abelian varieties.

Birkar: compact moduli exist for polarised log Calabi-Yau and log Fano pairs.

The latter is based on moduli of stable pairs.

Singularities and complements

Complement theory: Shokurov, Prokhorov studied $|-nK_X|$ relatively over $z \in Z$.

An n-complement over z is as $K_X + \Delta$ such that over z

- (X, Δ) is lc,
- $n(K_X + \Delta) \sim 0$.

Birkar: for Fano type fibrations, n-complements exist for n depending only on $\dim X$.

Conjecture (Shokurov): for Fano type ϵ-lc fibrations with $\epsilon > 0$, klt n-complements exist for n depending only on $\epsilon, \dim X$.

Conjecture (Shokurov): if $(X, B) \to Z$ is ϵ-lc and of Fano type, then (Z, B_Z) is δ-lc for δ depending only on $\epsilon, \dim X$.

These are all related to the ACC for mld's conjecture and ultimately to termination of flips.

Group theory

Let C_n be the group of field automorphisms of $\mathbb{C}(t_1, \ldots, t_n)$ fixing $\mathbb{C}$.

$C_n \simeq$ the set of all birational maps $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^n_{\mathbb{C}}$. Composition of maps gives it the group structure.

A birational map $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^n_{\mathbb{C}}$ is a map defined by homog polynomials of same degree, which has an inverse.

Example: $\phi \colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ defined by

$$(a_0 : a_1 : a_2) \mapsto (a_1 a_2 : a_0 a_2 : a_0 a_1)$$

is a birational map.

C_n contains $\mathrm{PGL}_{n+1}(\mathbb{C}) =$ set of $(n+1) \times (n+1)$ invertible matrices.

Elements of $\mathrm{PGL}_{n+1}(\mathbb{C})$ give isomorphisms $\mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$.

Group theory

$C_1 = \mathrm{PGL}_2(\mathbb{C})$.

C_2 is generated by $\mathrm{PGL}_3(\mathbb{C})$ and ϕ defined above.

Question (Serre): Is C_n Jordan? i.e. is there $h \in \mathbb{N}$ such that for any finite subgroup G of C_n, there is a normal abelian subgroup H of G of index at most h?

Serre: yes for $n = 2$.

Prokhorov and Shramov studied Serre's question using birational algebraic geometry.

Prokhorov-Shramov + Birkar's boundedness: C_n is Jordan.

Blanc, Lamy, Zimmermann: C_n is not simple, $n \geq 3$. The case $n = 2$ was done earlier by Cantat, Lamy.

Rationality and rigidity

Question: Which Fano varieties are rational?

Question: Which smooth hypersurfaces $X_d \subset \mathbb{P}^n$ are rational?

Clemens, Griffiths: $X_3 \subset \mathbb{P}^4$ is non-rational.

Recall: Iskovkikh-Manin showed $X_4 \subset \mathbb{P}^4$ is non-rational.

Pukhlikov, Cheltsov, Corti, de Fernex, Ein, Mustaţă: $X_n \subset \mathbb{P}^n$ is birationally rigid, $n \geq 4$.

Kollár: a very general $X_d \subset \mathbb{P}^n$ is non-rational, $d \geq \lceil (n+2)/3 \rceil$ and $n \geq 4$.

Voisin, Colliot-Thélène, Pirutka introduced new ideas.

Totaro: a very general $X_d \subset \mathbb{P}^n$ is not stably rational, $d \geq \lceil (n+1)/3 \rceil$ and $n \geq 4$.

Schreieder: a very general $X_d \subset \mathbb{P}^n$ is not stably rational, $d \geq \log_2 n + 2$ and $n \geq 4$.

Arithmetic geometry

For a variety X over a field k, let $X(k) = \{k\text{-rational points on } X\}$.

Esnault: X smooth Fano, k a finite field, then $X(k) \neq \emptyset$.

Graber-Harris-Starr: X rationally connected, k function field of a curve, $X(k) \neq \emptyset$.

Mordell-Weil: X abelian variety, k number field, $X(k)$ is a finitely generated abelian group.

Conjecture (Lang, Campana, etc):
If X is a smooth Fano or Calabi-Yau variety over a number field k, then there is a finite extension $k \subseteq k'$ such that $X(k')$ is dense.

Fibrations are even harder to understand.

Conjecture:
If $X \to \mathbb{P}^2$ is a conic bundle over a number field k, then there is a finite extension $k \subseteq k'$ such that $X(k')$ is dense.

Metrics and K-stability

Siu-Yau proved Frankel Conjecture: a compact Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to some $\mathbb{P}^n$.

Yau proved Calabi conjecture: on a smooth Calabi-Yau variety there is a Kähler metric with vanishing Ricci curvature.

Chen, Donaldson, Sun and Tian proved Yau-Tian-Donaldson conjecture: a smooth Fano X admits a Kähler-Einstein metric if and only if X is K-polystable.

Blum, Fujita, Li, Liu, Odaka, Xu, Zhuang and others have tried to construct moduli of K-stable Fano's after work of Yau, Tian, Donaldson, Chen, Sun and others.

Mirror Symmetry

The Mirror Symmetry phenomenon emerged from mathematical physics.

Example (Batyrev): $P \subset \mathbb{R}^3$ a reflexive lattice polytope, $\check{P}$ its dual, Y_P and $Y_{\check{P}}$ toric Fano varieties; $X \in |-K_{Y_P}|$ are mirror to $\check{X} \in |-K_{Y_{\check{P}}}|$. This includes the famous quintic example.

Conjecture (Strominger-Yau-Zaslow Mirror Symmetry):
a smooth Calabi-Yau X is expected to have a mirror Calabi-Yau $\check{X}$ s.t.

- X and $\check{X}$ have special Lagrangian torus fibrations $f\colon X \to B$ and $\check{f}\colon \check{X} \to B$,
- smooth fibres of $f, \check{f}$ over each $b \in B$ are dual tori, and
- transforms on fibres interchange complex-geometric data on X with symplectic-geometric data on $\check{X}$.

Conjecture (Kontsevich, Homological Mirror Symmetry):
a smooth Calabi-Yau variety X is expected to have a mirror Calabi-Yau $\check{X}$ s.t.

$$\mathcal{D}^b(X) \simeq \mathcal{F}(\check{X}).$$

Other topics

Derived categories: this is closely related to MMP. Conjecture: K-equivalence implies D-equivalence.

Moduli of stable sheaves. Bridgeland stability.