Geometry of polarised varieties

Caucher Birkar

Cambridge University

ZAG Seminar, 4 June 2020

Polarised varieties

We work over an algebraically closed field k of char zero.

Assume X is a normal projective variety, N a nef and big (eg ample) integral divisor.

Questions:

- What can we say about the geometry of X, N?
- For which $m \in \mathbb{N}$, the linear system $|mN|$ defines a birational map?
- Under what conditions such X, N form a bounded family?
- Under what conditions is there a moduli space for such X, N?

In this talk we explore these questions.

Singularities

A **pair** (X, B) consists of a normal variety X and a boundary divisor B with coefficients in $[0, 1]$.

Singularities of (X, B) are defined by taking a log resolution $\phi\colon W \to X$ and writing $K_W + B_W = \phi^*(K_X + B)$.

We say (X, B) is **ϵ-lc** if every coefficient of B_W is $\leq 1 - \epsilon$.

Lc means 0-lc. Klt means ϵ-lc for some $\epsilon > 0$.

Examples:

- lc: X a smooth variety and B a simple normal crossing divisor.
- lc: X a smooth surface and B a nodal curve (but B cuspidal not lc).
- lc not klt: X cone over elliptic curve.
- $\frac{2}{n}$-lc: X cone over rational curve of degree n.

Birational boundedness of linear systems

For X, N as above, the linear system $|mN|$ gives a rational map $\phi_{|mN|}: X \dashrightarrow \mathbb{P}^h$.

Question: Is there a bounded m such that $\phi_{|mN|}$ is birational?

In dimension one: there is no bounded m that works for all X, N.

We need to impose some conditions to get reasonable results.

In practice this means get K_X involved.

There has been extensive studies when $N = K_X$ or $N = -K_X$ is ample.

Hacon-McKernan-Xu, 2012: If X is lc and $N = K_X$ is ample, then $\phi_{|mN|}$ is birational for some m depending on $\dim X$.

B, 2016: If X is ϵ-lc with $\epsilon > 0$ and $N = -K_X$ is ample, then $\phi_{|mN|}$ is birational for some m depending on $\dim X, \epsilon$.

Theorem

Assume

- X *is projective* ϵ*-lc with* $\epsilon > 0$*,*
- N *is integral nef and big,*
- $N - K_X$ *is pseudo-effective.*

Then $\phi_{|mN|}$ *is birational for some* m *depending only on* $\dim X, \epsilon$*.*

Applying this to the case $N = -K_X$, we get a new proof of the result mentioned before.

The ϵ-lc condition cannot be removed.

For example, if X is klt Fano and $N = -K_X$, in general there is no bounded m such that $\phi_{|mN|}$ is birational.

We can replace ϵ-lc with klt and additional conditions.

Theorem

Assume

- *X is projective klt,*
- *N is integral nef and big,*
- *$N - K_X$ and K_X are pseudo-effective.*

Then $\phi_{|mN|}$ is birational for some m depending only on $\dim X$.

Applying each theorem we get:

Corollary

Assume

- *X is projective klt Calabi-Yau,*
- *N is integral, big.*

Then $\phi_{|mN|}$ is birational for some m depending only on $\dim X$.

By Calabi-Yau we simply mean $K_X \equiv 0$.

Remarks:

- The theorems and the corollary hold for pairs.
- $\phi_{|mN+L|}$ and $\phi_{|mN+L+K_X|}$ are birational for any integral pseudo-effective L.
- This birationality holds for any $m' \geq m$.
- Integrality of N can be replaced by: $N = E + R$ where E is integral pseudo-effective and coeffs of $R \geq \delta$ for fixed $\delta > 0$.

Previous known cases of the corollary:

smooth dim 2, Reider, 1988,

terminal dim 3, Jiang, using Riemann-Roch, 2015,

dim 3 special cases, Fukuda, 1991, and Oguiso-Peternell, 1995,

smooth irreducible symplectic varieties, $N \geq 0$, Kapustka, Mongardi, Pacienza, Pokora, 2019.

Boundedness of polarised varieties

Question: Given a projective X and integral nef and big N, under what conditions X, N are bounded?

Theorem

Let $d \in \mathbb{N}$ and $\epsilon, \delta, v \in \mathbb{R}^{>0}$. Assume

- *(X, B) is projective ϵ-lc of dim $= d$,*
- *$coeff(B) \subset \{0\} \cup [\delta, \infty)$,*
- *$K_X + B$ is nef,*
- *N is integral nef and big,*
- *$\mathrm{vol}(K_X + B + N) \leq v$.*

Then such $(X, \mathrm{Supp}\, B)$ form a bounded family.
If $N \geq 0$, then such $(X, \mathrm{Supp}(B + N))$ form a bounded family.

Rough idea of proof: for simplicity, assume $B = 0$;

Use birational boundedness to find bounded $m, l \in \mathbb{N}$ and $M \in |mK_X + lmN|$.

Show $(X, \operatorname{Supp} M)$ is birationally bounded.

Show (X, tM) is lc for some $t \in \mathbb{Q}^{>0}$ bounded away from zero.

From these we can derive boundedness of $(X, \operatorname{Supp} M)$.

If $N \geq 0$, we can assume $M \geq N$, so $(X, \operatorname{Supp} N)$ is bounded.

Corollary

Let $d \in \mathbb{N}$, $v \in \mathbb{R}^{>0}$, and $\Phi \subset [0, 1]$ be a DCC set of rational numbers. Assume

- *(X, B) is a projective klt Calabi-Yau pair of dim $= d$,*
- *coeff$(B) \subset \Phi$,*
- *N is integral nef and big,*
- $\operatorname{vol}(N) \leq v$.

Then such $(X, \operatorname{Supp} B)$ form a bounded family.
If $N \geq 0$, then such $(X, \operatorname{Supp}(B + N))$ form a bounded family.

Relaxing the nef and bigness of N to only big, X is birationally bounded as we can apply the corollary to the minimal model of N.

In the Calabi-Yau case we can go further.

A **polarised slc Calabi-Yau pair** consists of a projective slc Calabi-Yau pair (X, B) and an ample integral divisor $N \geq 0$ such that $(X, B + uN)$ is slc for some real number $u > 0$.

Theorem

Let $d \in \mathbb{N}$, $v \in \mathbb{Q}^{>0}$, and $\Phi \subset [0, 1]$ be a DCC set of rational numbers. Assume

- *$(X, B), N$ is a polarised slc Calabi-Yau pair of dim d,*
- *$\mathit{coeff}(B) \subset \Phi$,*
- *$\mathrm{vol}(N) = v$.*

Then such $(X, \mathrm{Supp}(B + N))$ form a bounded family.

Bounding lc thresholds again plays a key role in the proof of this.

We show $(X, B + tN)$ is slc for some $t \in \mathbb{Q}^{>0}$ depending only on d, v, Φ.

Then we use a result of Hacon-M$^{\text{c}}$Kernan-Xu.

Moduli of polarised Calabi-Yau pairs

Combining the above boundedness results with moduli theory of stable pairs shows that there exists a projective coarse moduli space for polarised slc Calabi-Yau pairs.

First we say a few words about stable pairs.

The subject has a long history, e.g. Deligne-Mumford, Kollár-Shepherd-Barron, Alexeev, Kollár, etc (with input from many others). Today we follow Kollár.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A **stable pair** is a connected projective pure dim slc pair (X, Δ) with $K_X + \Delta$ ample.

If $\dim X = d$, $\Delta = cD$ for some effective integral divisor D, and $\mathrm{vol}(K_X + \Delta) = v$, then we say (X, Δ) is a (d, c, v)-stable pair.

It takes more work to define (d, c, v)-stable families $(X, \Delta) \to S$. Roughly this is a flat projective family with (d, c, v)-stable log fibres.

Next we treat moduli of polarised Calabi-Yau pairs following Alexeev, Hacking, Laza, DeVleming, Kollár-Xu, etc.

A (d, c, v)**-polarised slc Calabi-Yau pair** is defined by the data:

- $(X, B + uN)$ is a stable pair of dimension d, for some $u \in \mathbb{Q}^{>0}$,
- $B = cD$ for some integral divisor $D \geq 0$,
- $N \geq 0$ is an integral divisor,
- $K_X + B \sim_{\mathbb{Q}} 0$,
- $\mathrm{vol}(N) = v$.

A family of (d, c, v)**-polarised Calabi-Yau pairs** over a reduced k-scheme S is defined by the data:

- $(X, B + uN) \to S$ is a stable family with d-dim fibres, for some $u \in \mathbb{Q}^{>0}$,
- $B = cD$ for some relative Mumford divisor $D \geq 0$,
- $N \geq 0$ is a relative Mumford divisor,
- $K_{X/S} + B \sim_{\mathbb{Q}} 0/S$,
- for any fibre X_s, $\mathrm{vol}(N|_{X_s}) = v$.

Define the moduli functor $\mathcal{PCY}_{d,c,v}$ on reduced k-schemes by setting

$$\mathcal{PCY}_{d,c,v}(S) = \{\text{families of } (d,c,v)\text{-polarised Calabi-Yau pairs over } S, \text{ up to isomorphism over } S.\}$$

Theorem

The functor $\mathcal{PCY}_{d,c,v}$ has a projective coarse moduli space.

Recently, Kollár-Xu proved projectivity of irreducible components of this moduli space.

There is related work of Alexeev, etc, on moduli of abelian varieties and K3 surfaces.

For the proof: given a (d,c,v)-polarised Calabi-Yau pair (X,B), N, applying results above, $(X, B+tN)$ is a bounded stable pair, for some fixed $t \in \mathbb{Q}^{>0}$. Thus we can embed X into a fixed $\mathbb{P}^n$.

Then we show existence of the moduli space of embedded polarised Calabi-Yau pairs using existence of moduli of embedded stable pairs (by Kollár), and then take quotient.

Moduli of polarised Fano pairs

Let $d \in \mathbb{N}$ and $c, a, v \in \mathbb{Q}^{>0}$.

A (d, c, a, v)**-polarised slc Fano pair** is defined by the data:

- $(X, B + uN)$ is a stable pair of dimension d for some rational number $u > a$,
- $B = cD$ for some integral divisor $D \geq 0$,
- $N \geq 0$ is an integral divisor,
- $K_X + B + aN \sim_{\mathbb{Q}} 0$,
- $\mathrm{vol}(N) = v$.

Families of (d, c, a, v)-polarised slc Fano pairs and the corresponding functor $\mathcal{PF}_{d,c,a,v}$ can be defined similarly.

Theorem

The functor $\mathcal{PF}_{d,c,a,v}$ has a projective coarse moduli space.

Special cases were treated in the literature.

Hacking treated the case X smoothable to $\mathbb{P}^2$,

Deopurkar-Han the case smoothable to $\mathbb{P}^1 \times \mathbb{P}^1$, and

DeVleming the case smoothable to a fixed smooth Fano.

Rough sketch of proof of birational boundedness

Theorem

Assume

- X *is projective* ϵ*-lc,*
- N *is integral nef and big,*
- $N - K_X$ *is pseudo-effective.*

Then $\phi_{|mN|}$ *is birational for some* m *depending only on* $\dim X, \epsilon$.

Enough to find bounded m so that for general $x, y \in X$ (possibly switching them), there is $0 \leq \Delta \sim_{\mathbb{Q}} mN - K_X$ such that (X, Δ) is lc at x and $\{x\}$ is an isolated lc centre but (X, Δ) is not klt at y.

Because then using vanishing theorems, we get $\alpha \in H^0(X, mN)$ with $\alpha(x) \neq 0$ but $\alpha(y) = 0$, and vice versa.

This basic strategy has been used numerous times by many for a long time.

To proceed, we reduce to the case when $N - K_X$ and $N + K_X$ are big: this relies on BAB.

Pick m such that $\mathrm{vol}(mN - K_X) > (2d)^d$ (initially we take m minimal).

There is $\leq \Delta \sim_{\mathbb{Q}} mN - K_X$ such that (X, Δ) is lc at x with a unique lc centre G but (X, Δ) is not klt at y.

If always $\dim G = 0$, then $\phi_{|mN|}$ is birational. Eventually need to bound m from above.

Hard part: if $\dim G > 0$, show $\mathrm{vol}(mN|_G)$ is bounded from below in order to replace G and decrease dimension.

Can write

$$mN|_F \sim_{\mathbb{Q}} (K_X + \Delta)|_F \sim_{\mathbb{Q}} K_F + \Theta_F + P_F$$

where F is normalisation of G, Θ_F is a boundary, and P_F is big.

Show F is birational to a bounded F'.

If $(F, \Theta_F + P_F)$ has bad singularities, then we can bound $\mathrm{vol}(mN|_G)$ by comparing singularities with those on F'.

This allows to reduce to the case when $(F, \Theta_F + P_F)$ is $\frac{\epsilon}{2}$-lc.

Key point: in this case we can show $N|_F$ is integral up to a bounded multiple. Hence we can assume $mN|_F$ is integral.

Since $mN|_F - K_F$ is big, applying induction on dimension, $\phi_{|mN|_F|}$ is birational.

Therefore, $\mathrm{vol}(mN|_G) \geq 1$.

Eventually we need to show m is bounded.

Remarks:

- If $N = K_X$ or $N = -K_X$, we do not need to use BAB.
- If instead of N integral we assume $N = E + R$ where E is integral pseudo-effective, coefficients of R are $\geq \delta$, can ensure $N|_F = E|_F + R|_F$ has similar properties.

Sketch of proof of boundedness of polarised Calabi-Yau pairs

Theorem

Let $d \in \mathbb{N}$, $v \in \mathbb{Q}^{>0}$, and $\Phi \subset [0,1]$ be a DCC set of rational numbers. Assume

- $(X, B), N$ *is a polarised slc Calabi-Yau pair of dim* d,
- *coeff*$(B) \subset \Phi$,
- $\operatorname{vol}(N) = v$.

Then such $(X, \operatorname{Supp}(B+N))$ *form a bounded family.*

Enough to show $(X, B+tN)$ is slc for fixed $t \in \mathbb{Q}^{>0}$, by Hacon-M^{c}Kernan-Xu.

Taking normalisation and picking an irreducible component, can assume (X, B) is lc. But now $\operatorname{vol}(N) \leq v$.

Replace (X, B) with a $\mathbb{Q}$-factorial dlt model, N with its pullback: N is still integral.

Pick ϵ sufficiently small.

Extract all divisors D with $a(D, X, 0) \leq \epsilon$, say via $Y \to X$.

Replace X with Y and $K_X + B, N$ with their pullback: N is still integral.

Now X is ϵ-lc, N is integral nef and big, and $N - K_X \sim_{\mathbb{Q}} N + B$ is big.

Apply birational boundedness to find bounded m such that $\phi_{|mN|}$ is birational.

This implies $(X, \mathrm{Supp}(B + N))$ is birationally bounded.

Comparing singularities on X and the bounded model ensures t exists.