ON SYMMETRIC INTERSECTING FAMILIES

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Abstract. A family of sets is said to be symmetric if its automorphism group is transitive, and intersecting if any two sets in the family have nonempty intersection. Our purpose here is to study the following question: for \( n, k \in \mathbb{N} \) with \( k \leq n/2 \), how large can a symmetric intersecting family of \( k \)-element subsets of \( \{1, 2, \ldots, n\} \) be? As a first step towards a complete answer, we prove that such a family has size at most
\[
\exp\left(-\frac{c(n - 2k) \log n}{k (\log n - \log k)}\right) \binom{n}{k},
\]
where \( c > 0 \) is a universal constant. We also describe various combinatorial and algebraic approaches to constructing such families.

1. Introduction

A family of sets is said to be intersecting if any two sets in the family have nonempty intersection, and uniform if all the sets in the family have the same size. In this paper, we study uniform intersecting families. The most well-known result about such families is the Erdős–Ko–Rado theorem [11].

Theorem 1.1. Let \( n, k \in \mathbb{N} \) with \( k \leq n/2 \). If \( \mathcal{A} \) is an intersecting family of \( k \)-element subsets of \( \{1, 2, \ldots, n\} \), then \( |\mathcal{A}| \leq \binom{n-1}{k-1} \). Furthermore, if \( k < n/2 \), then equality holds if and only if \( \mathcal{A} \) consists of all the \( k \)-element subsets of \( \{1, 2, \ldots, n\} \) which contain some fixed element \( i \in \{1, 2, \ldots, n\} \).

Over the last fifty years, many variants of this theorem have been obtained. A common feature of many of these variants is that the extremal families in these results are highly asymmetric; this is the case, for example, in the Erdős–Ko–Rado theorem itself, in the Hilton–Milner theorem [16], and in Frankl’s generalisation [12] of these results. It is therefore natural to ask what happens to the maximum possible

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size of a uniform intersecting family when one imposes a symmetry requirement on
the family.

To make the idea of a ‘symmetric’ family precise, we need a few definitions. For
a positive integer \( n \in \mathbb{N} \), we denote the set \( \{1, 2, \ldots, n\} \) by \([n]\). We write \( S_n \) for
the symmetric group on \([n]\) and \( \mathcal{P}_n \) for the power-set of \([n]\). For a permutation
\( \sigma \in S_n \) and a set \( x \subset [n] \), we write \( \sigma(x) \) for the image of \( x \) under \( \sigma \), and if \( \mathcal{A} \subset \mathcal{P}_n \),
we write \( \sigma(\mathcal{A}) = \{\sigma(x) : x \in \mathcal{A}\} \). We define the automorphism group of a family \( \mathcal{A} \subset \mathcal{P}_n \) by

\[
\text{Aut}(\mathcal{A}) = \{\sigma \in S_n : \sigma(\mathcal{A}) = \mathcal{A}\}.
\]

We say that \( \mathcal{A} \subset \mathcal{P}_n \) is symmetric if \( \text{Aut}(\mathcal{A}) \) is a transitive subgroup of \( S_n \), i.e., if for all \( i, j \in [n] \), there exists a permutation \( \sigma \in \text{Aut}(\mathcal{A}) \) such that \( \sigma(i) = j \).

For a pair of integers \( n, k \in \mathbb{N} \) with \( k \leq n \), let \( [n]^{(k)} \) denote the family of all \( k \)-element subsets of \([n]\). In this paper, we will be concerned with estimating the quantity

\[
s(n, k) = \max\{|\mathcal{A}| : \mathcal{A} \subset [n]^{(k)} \text{ such that } \mathcal{A} \text{ is symmetric and intersecting}\}.
\]

Of course, if \( k > n/2 \), then \( [n]^{(k)} \) itself is a symmetric intersecting family, so \( s(n, k) = \binom{n}{k} \); we may therefore restrict our attention to the case where \( k \leq n/2 \). We will particularly be interested in determining when a symmetric uniform intersecting family must be significantly smaller than the extremal families (of the same uniformity) in the Erdős–Ko–Rado theorem. A more precise formulation of this question is as follows.

**Problem 1.2.** For which \( k = k(n) \leq n/2 \) is \( s(n, k) = o\left(\left(\frac{n-1}{k-1}\right)\right) \)?

In this paper, by utilising a well-known sharp threshold result of Friedgut and
the second author [14], together with an estimate of Friedgut [13], we prove the following.

**Theorem 1.3.** There exists a universal constant \( c > 0 \) such that for any \( n, k \in \mathbb{N} \) with \( k \leq n/2 \), we have

\[
s(n, k) \leq \exp\left(-\frac{c(n - 2k) \log n}{k(\log n - \log k)}\right) \binom{n}{k}.
\]

We also give a construction showing that Theorem 1.3 is sharp up to the value of \( c \) in the regime where \( k/n \) is bounded away from zero. This construction, in conjunction with Theorem 1.3, provides a complete solution to Problem 1.2.
**Proposition 1.4.** If \( k = k(n) \leq n/2 \), then as \( n \to \infty \), \( s(n,k) = o\left(\binom{n-1}{k-1}\right) \) if and only if \( n = o\left( (n-2k) \log n \right) \).

This paper is organised as follows. We give the proof of Theorem 1.3 in Section 2. In Section 3, we describe a way to construct large symmetric intersecting families in the regime where \( k \) is comparable to \( n \) and deduce Proposition 1.4. In Section 4, we study some questions about the existence of nonempty, symmetric intersecting families and we describe various constructions of such families in the regime where \( k \) is comparable to \( \sqrt{n} \). We finally conclude in Section 5 with a discussion of some open problems and related work.

2. Upper bounds

We first describe briefly the notions and tools we will need for the proof of Theorem 1.3. In what follows, all logarithms are to the base \( e \).

We begin with the following simple observation which may be found in [5], for example; we include a proof for completeness.

**Proposition 2.1.** For all \( n,k \in \mathbb{N} \) with \( 1 < k \leq \sqrt{n} \), we have \( s(n,k) = 0 \).

**Proof.** The proposition follows from a simple averaging argument. Indeed, let \( k \leq \sqrt{n} \), suppose for a contradiction that \( A \subset [n]^k \) is a nonempty, symmetric intersecting family, and let \( x \in A \). If we choose \( \sigma \in \text{Aut}(A) \) uniformly at random, then since \( \text{Aut}(A) \) is transitive, we have

\[
\mathbb{E}[|x \cap \sigma(x)|] = \frac{k^2}{n} \leq 1.
\]

Since \( |x \cap \text{Id}(x)| = k > 1 \), there must exist a permutation \( \sigma \in \text{Aut}(A) \) such that \( x \cap \sigma(x) = \emptyset \), contradicting the fact that \( A \) is intersecting. \( \square \)

For \( 0 \leq p \leq 1 \), we write \( \mu_p \) for the \( p \)-biased measure on \( \mathcal{P}_n \), defined by

\[
\mu_p(\{x\}) = p^{|x|}(1-p)^{n-|x|} \quad \forall x \subset [n].
\]

We say that a family \( \mathcal{F} \subset \mathcal{P}_n \) is *increasing* if it is closed under taking supersets, i.e., if \( x \in \mathcal{F} \) and \( x \subset y \), then \( y \in \mathcal{F} \). It is easy to see that if \( \mathcal{F} \subset \mathcal{P}_n \) is increasing, then \( p \mapsto \mu_p(\mathcal{F}) \) is a monotone non-decreasing function on \([0,1]\). For a family \( \mathcal{F} \subset \mathcal{P}_n \), we write \( \mathcal{F}^\uparrow \) for the smallest increasing family containing \( \mathcal{F} \); in other words, \( \mathcal{F}^\uparrow = \{ y \subset [n] : x \subset y \text{ for some } x \in \mathcal{F} \} \).
We need the following fact, which allows one to bound from above the size of a family $F \subset [n]^{(k)}$ in terms of $\mu_p(F^\uparrow)$, where $p \approx k/n$; this was proved in a slightly different form by Friedgut [13]. We provide a proof for completeness.

**Lemma 2.2.** Let $n, k \in \mathbb{N}$ and suppose that $0 < p, \phi < 1$ satisfy

$$p \geq \frac{k}{n} + \frac{\sqrt{2n \log(1/\phi)}}{n}.$$ 

Then for any family $F \subset [n]^{(k)}$, we have

$$\mu_p(F^\uparrow) > (1 - \phi) \frac{|F|}{\binom{n}{k}}.$$

**Proof.** Let $\delta = |F|/\binom{n}{k}$ and let $X$ be a binomial random variable with distribution $\text{Bin}(n, p)$. First, for each $l \geq k$, the local LYM inequality implies that

$$\frac{|F^\uparrow \cap [n]^{(l)}|}{\binom{n}{l}} \geq \frac{|F|}{\binom{n}{k}} = \delta.$$ 

Now, for any $\eta > 0$, it follows from a standard Chernoff bound that

$$\Pr(X < (1 - \eta)np) < \exp(-\eta^2 np/2).$$

Hence,

$$\mu_p(F^\uparrow) \geq \sum_{l=k}^{n} p^l (1 - p)^{n-l} \binom{n}{l} \delta = \Pr(X \geq k) \delta > (1 - \phi) \delta. \quad \square$$

We will also require the following sharp threshold result due to Friedgut and the second author [14].

**Theorem 2.3.** There exists a universal constant $c_0 > 0$ such that the following holds for all $n \in \mathbb{N}$. Let $0 < p, \varepsilon < 1$ and let $F \subset \mathcal{P}_n$ be a symmetric increasing family. If $\mu_p(F) > \varepsilon$, then $\mu_q(F^\uparrow) > 1 - \varepsilon$, where

$$q = \min \left\{ 1, p + c_0 \left( \frac{p \log(1/p) \log(1/\varepsilon)}{\log n} \right) \right\}. \quad \square$$

The idea of the proof of Theorem 1.3 is as follows. Let $A \subset [n]^{(k)}$ be a symmetric intersecting family. We first use Lemma 2.2 to bound $|A|/\binom{n}{k}$ from above in terms of $\mu_p(A^\uparrow)$, where $p \approx k/n$; we then use Theorem 2.3, together with the simple fact that $\mu_{1/2}(A^\uparrow) \leq 1/2$, to bound $\mu_p(A^\uparrow)$, and hence $|A|$, from above.

We remark that the strategy of ‘approximating’ the uniform measure on $[n]^{(k)}$ with the $p$-biased measure $\mu_p$, where $p \approx k/n$, is known to be useful in the study of
uniform intersecting families. Indeed, Friedgut [13] obtained a ‘stability’ result on the structure of ‘large’ $t$-intersecting families, for $t \geq 1$, by first using Lemma 2.2 (in a slightly different form) to approximate the uniform measure $|\mathcal{F}|/\binom{n}{k}$ of a family $\mathcal{F} \subset [n]^{(k)}$ in terms of the $p$-biased measure $\mu_p(\mathcal{F}^\uparrow)$, where $p \approx k/n$, and by then working with the $p$-biased measure. This approximation strategy was also used by Dinur and Friedgut [8] to obtain several results on the approximate containment of large uniform intersecting families inside ‘juntas’, and recently by the first author, Keller and Lifshitz [9] to obtain stability results on a number of Erdős–Ko–Rado type problems.

**Proof of Theorem 1.3.** Let $n, k \in \mathbb{N}$ with $k \leq n/2$, let $A \subset [n]^{(k)}$ be a symmetric intersecting family, and set $\delta = |A|/\binom{n}{k}$.

In the light of Proposition 2.1, we may suppose without loss of generality that $k > \sqrt{n}$. By the Erdős–Ko–Rado theorem, we have $\delta \leq k/n \leq 1/2$, so we may also assume (by choosing $c > 0$ sufficiently small) that $k \leq n/2 - 10n/\log n$.

First, applying Lemma 2.2 with $p = k/n + \sqrt{(2\log 2)n/n}$ and $\phi = 1/2$, we see that

$$
\mu_p(A^\uparrow) > \frac{\delta}{2}.
$$

Next, since $A$ is symmetric, so is $A^\uparrow$. We may therefore apply Theorem 2.3 with $\varepsilon = \delta/2$ to deduce that $\mu_q(A^\uparrow) > 1/2$, where

$$
q = \min \left\{ 1, p + c_0 \left( \frac{p \log(1/p) \log(2/\delta)}{\log n} \right) \right\}.
$$

Since $A^\uparrow$ is increasing, the function $r \mapsto \mu_r(A^\uparrow)$ is monotone non-decreasing on $[0, 1]$. Also, since $A$ is intersecting, so is $A^\uparrow$, and therefore $\mu_{1/2}(A^\uparrow) \leq 1/2$. Now, as $\mu_{1/2}(A^\uparrow) \leq 1/2$ and $\mu_q(A^\uparrow) > 1/2$, the monotonicity of $r \mapsto \mu_r(A^\uparrow)$ implies that

$$
p + c_0 \left( \frac{p \log(1/p) \log(2/\delta)}{\log n} \right) > \frac{1}{2}.
$$

Rearranging this inequality, we see that

$$
\delta < 2 \exp \left( -\frac{(1-2p) \log n}{2c_0 p \log(1/p)} \right) \leq \exp \left( -\frac{c(n-2k) \log n}{k \log n - \log k} \right),
$$

where the last inequality above holds for some universal constant $c > 0$ provided $\sqrt{n} < k \leq n/2 - 10n/\log n$; this proves the theorem. \hfill \Box
3. Lower bounds for large $k$

In this section, we give a construction showing that Theorem 1.3 is sharp up to the value of the constant $C$ in the exponent for many choices of $k = k(n)$.

Given $n, k \in \mathbb{N}$ with $k \leq n$, we identify $[n]$ with $\mathbb{Z}_n$, we identify a subset $S \subset \mathbb{Z}_n$ with its characteristic vector $\chi_S \in \{0, 1\}^\mathbb{Z}_n$, and we define $\mathcal{F}(n, k)$ to be the family of all $k$-element subsets $S \subset \mathbb{Z}_n$ such that the longest run of consecutive ones in $\chi_S$ is longer than the longest run of consecutive zeros in $\chi_S$. Slightly less formally, we take $\mathcal{F}(n, k)$ to consist of all the cyclic strings of $n$ zeros and ones which contain exactly $k$ ones and in which the longest run of consecutive ones is longer than the longest run of consecutive zeros.

It is clear that $\mathcal{F}(n, k)$ is symmetric. It is also easy to check that $\mathcal{F}(n, k)$ is intersecting. Indeed, given $S, T \in \mathcal{F}(n, k)$, suppose without loss of generality that the longest run of consecutive ones in $S$ is at least as long as that in $T$. Choose a run of consecutive ones in $S$ of the maximum length; these cannot be all zeros in $T$ because otherwise $T$ would have a longer run of consecutive ones than $S$. Therefore, $S \cap T \neq \emptyset$.

We note that the non-uniform case of this construction, i.e., the family of all cyclic strings of $n$ zeros and ones in which the longest run of consecutive ones is longer than the longest run of consecutive zeros, shows that the Kahn–Kalai–Linial theorem [17] cannot be improved by more than a constant factor for intersecting families; see [18] for more details.

When $k/n \geq \varepsilon$ for some fixed constant $\varepsilon > 0$, Theorem 1.3 implies that

$$s(n, k) \leq \exp\left(-\frac{\delta(n - 2k) \log n}{n}\right) \binom{n}{k}$$

for some constant $\delta > 0$ depending on $\varepsilon$ alone. The following lower bound for $|\mathcal{F}(n, k)|$ shows that Theorem 1.3 is sharp up to the constant factor in the exponent when $k/n$ is bounded away from zero.

**Lemma 3.1.** For each $\varepsilon > 0$, there exists $C > 0$ such that for any $n, k \in \mathbb{N}$ with $\varepsilon \leq k/n \leq 1/2$, we have

$$|\mathcal{F}(n, k)| \geq \exp\left(-\frac{C(n - 2k) \log n}{n}\right) \binom{n}{k}.$$  

In fact, we will prove the following stronger statement.
**Lemma 3.2.** If $k = k(n) \leq n/2$ is such that $\sqrt{n \log n} = o(k)$ as $n \to \infty$, then

$$|\mathcal{F}(n, k)| \geq \exp\left(-(1 + o(1))\left(\frac{\log(n - k) - \log k}{\log n - \log(n - k)}\right) \log n\right)\left(\frac{n}{k}\right).$$

To prove Lemma 3.2, we need the following.

**Lemma 3.3.** Let $k < n$. The number of cyclic strings of $n$ zeros and ones with $k$ ones and a run of consecutive zeros of length at least $l$ is at most $\frac{1}{4}n^k$, provided

$$l \geq \frac{\log n + 2 \log 2}{\log n - \log(n - k)}.$$ 

**Proof.** The number of such strings is at most $n\binom{n-l}{k}$, since (possibly overcounting) there are $n$ choices for the position of the run of $l$ consecutive zeros, and then $\binom{n-l}{k}$ choices for the positions of the ones. We have

$$\frac{n\binom{n-l}{k}}{\binom{n}{k}} = \frac{n(n-k)(n-k-1)\ldots(n-k-l+1)}{n(n-1)\ldots(n-l+1)} \leq n\left(\frac{n-k}{n}\right)^l \leq \frac{1}{4},$$

provided $l \geq (\log n + 2 \log 2)/\left(\log n - \log(n - k)\right)$, as required. \hfill \Box

If $k \leq n/2$, then the number of cyclic strings of length $n$ with $k$ ones and a run of consecutive ones of length at least $l$ is at most the number of cyclic strings of length $n$ with $k$ ones and a run of consecutive zeros of length at least $l$, so the following is now immediate.

**Corollary 3.4.** Let $k \leq n/2$. The number of cyclic strings of length $n$ with $k$ ones and no run of $l$ consecutive zeros or ones is at least $\frac{1}{2}n^k$, provided

$$l \geq \frac{\log n + 2 \log 2}{\log n - \log(n - k)}. \hfill \Box$$

We are now ready to prove Lemma 3.2.

**Proof of Lemma 3.2.** Choose $l_0 \in \mathbb{N}$ such that

$$l_0 - 1 \geq \frac{\log(n - l_0 - 2) + 2 \log 2}{\log(n - l_0 - 2) - \log(n - k - 2)}.$$ \hfill (2)

Observe that $\mathcal{F}(n, k)$ contains all cyclic strings of length $n$ with $k$ ones, precisely one run of $l_0$ consecutive ones, all other runs of consecutive ones having length at
most $l_0 - 2$, and no run of $l_0$ consecutive zeros. We claim that if $l_0 < n/2$, then the number of such strings is at least
\[
\frac{n}{2} \binom{n - l_0 - 2}{k - l_0}.
\]
Indeed, there are $n$ choices for the position of the run of $l_0$ consecutive ones, and there must be a zero on each side of this run of ones. Now, there are at least $\frac{1}{2} \binom{n - l_0 - 2}{k - l_0}$ choices for the remainder of the cyclic string (by Corollary 3.4), since if we take a cyclic string of length $n - l_0 - 2$ which contains no run of $l_0 - 1$ consecutive ones or zeros, and then insert (at some point) a run of $l_0$ consecutive ones with a zero on either side into this string, then the resulting string has the desired property provided $l_0 < n/2$.

It is easily checked that if $\sqrt{n \log n} = o(k)$, then we may choose $l_0 \in \mathbb{N}$ satisfying (2) such that
\[
l_0 = (1 + o(1)) \frac{\log n}{\log n - \log(n - k)};
\]
we then have $l_0 < n/2$ and
\[
|\mathcal{F}(n, k)| \geq \frac{n}{2} \binom{n - l_0 - 2}{k - l_0} \geq \frac{n}{2} \binom{n - l_0 - 2}{k - l_0}
\]
\[
\geq \frac{n}{2} \binom{k - l_0 - 2}{n - l_0 - 2}^{l_0 + 2} \binom{n}{k}
\]
\[
= \exp \left( -(1 + o(1)) \frac{\log(n - k) - \log k}{\log n - \log(n - k)} \log n \right) \binom{n}{k},
\]
proving the lemma. \hfill \Box

We can now prove Lemma 3.1.

**Proof of Lemma 3.1.** Writing $\eta = (n - 2k)/2n$, we have $\eta \leq 1/2 - \epsilon$, and
\[
\frac{\log(n - k) - \log k}{\log n - \log(n - k)} = \frac{\log(1 + 2\eta) - \log(1 - 2\eta)}{\log 2 - \log(1 + 2\eta)} = \frac{4\eta + O_\epsilon(\eta^3)}{\log 2 - \log(1 + 2\eta)} = \Theta_\epsilon(\eta).
\]
Hence, it follows from Lemma 3.2 that
\[
|\mathcal{F}(n, k)| \geq \exp(-2C\eta \log n) \binom{n}{k}
\]
for some constant $C > 0$ depending on $\epsilon$ alone, as required. \hfill \Box

Proposition 1.4 is now easily established.
Proof of Proposition 1.4. First, recall from Proposition 2.1 that \( s(n, k) = 0 \) if \( k \leq \sqrt{n} \). Next, suppose that \( \sqrt{n} < k \leq n / \log n \). Then, by Theorem 1.3, we have

\[
  s(n, k) \leq \exp \left( - \frac{cn \log n}{2k(\log n - \log k)} \right) \binom{n}{k}
  \leq \exp \left( - \frac{c(\log n)^2}{2} \right) \binom{n}{k}
  = o \left( \frac{1}{\sqrt{n}} \binom{n}{k} \right) = o \left( k \binom{n}{k} \right) = o \left( \binom{n-1}{k-1} \right).
\]

Now, suppose that \( n / \log n \leq k \leq n / 4 \). Then, again by Theorem 1.3, we have

\[
  s(n, k) \leq \exp \left( - \frac{cn \log n}{2k(\log n - \log k)} \right) \binom{n}{k}
  \leq \exp \left( - \frac{2c \log n}{\log \log n} \right) \binom{n}{k}
  = o \left( \frac{1}{\log n} \binom{n}{k} \right) = o \left( \frac{k}{n} \binom{n}{k} \right) = o \left( \binom{n-1}{k-1} \right).
\]

Finally, suppose that \( n / 4 \leq k \leq n / 2 \) and let \( \zeta > 0 \). If \( n \geq \zeta (n - 2k) \log n \), then by applying Lemma 3.1 with \( \varepsilon = 1 / 4 \), we see that there exists a universal constant \( C > 0 \) such that

\[
  s(n, k) \geq e^{-C/\zeta} \binom{n}{k} \geq 2e^{-C/\zeta} \binom{n-1}{k-1}.
\] (3)

If, on the other hand, we have \( n \leq \zeta (n - 2k) \log n \), then it follows from (1) that there exists a universal constant \( \delta > 0 \) such that

\[
  s(n, k) \leq e^{-\delta/\zeta} \binom{n}{k} \leq 4e^{-\delta/\zeta} \binom{n-1}{k-1}.
\] (4)

The proposition is now immediate from (3) and (4). \( \square \)

4. Lower bounds for small \( k \)

Our aim in this section is to investigate the set

\[
  \mathcal{S} = \{ (n, k) \in \mathbb{N}^2 : s(n, k) > 0 \}.
\]

Along the way, we shall also describe some constructions of symmetric intersecting families which are larger than \( \mathcal{F}(n, k) \) for certain values of \( n \) and \( k \).
We begin by noting that if \( s(n, k) > 0 \), then \( s(n, l) > 0 \) for all \( l > k \); indeed, if \( \mathcal{A} \subset [n]^{(k)} \) is nonempty, symmetric and intersecting, then so is \( \{ y \in [n]^{(l)} : x \subset y \text{ for some } x \in \mathcal{A} \} \). In the light of this fact, for each \( n \in \mathbb{N} \), we define
\[
g(n) = \min\{ k \in \mathbb{N} : s(n, k) > 0 \}.
\]
Proposition 2.1 tells us that \( g(n) > \sqrt{n} \) for all \( n \geq 2 \); it is natural to ask whether this bound is asymptotically tight, so we pose the following question.

**Question 4.1.** Is it true that \( g(n) = (1 + o(1))\sqrt{n} \) for all \( n \in \mathbb{N} \)?

We begin with an easy upper bound for \( g(n) \). It is easy to check that \( \mathcal{F}(n, k) \neq \emptyset \) if and only if \( n \leq \lfloor k/2 \rfloor^2 + k \). This observation implies that
\[
g(n) \leq 2\sqrt{n}
\]
for all \( n \in \mathbb{N} \). To improve (5), we note a strong connection between the problem of determining \( g(n) \) and the problem of covering an Abelian group using a difference set. If \( G \) is a finite Abelian group and \( S \subset G \), we say that \( S \) is a difference cover for \( G \) if \( S - S = G \), i.e., if \( \{ i - j : i, j \in S \} = G \); we then define
\[
h(G) = \min\{|S| : S \text{ is a difference cover for } G\}.
\]
Note that if \( S \subset G \), then \( S \) is a difference cover for \( G \) if and only if the family of all the translates of \( S \) is an intersecting family of subsets of \( G \); this observation yields the following.

**Lemma 4.2.** For all \( n \in \mathbb{N} \), we have \( g(n) \leq h(\mathbb{Z}_n) \), with equality holding in the case where \( n \) is prime.

**Proof.** Let \( h = h(\mathbb{Z}_n) \) and write \( \mathbb{Z}_n^{(h)} \) for the family of \( h \)-element subsets of \( \mathbb{Z}_n \). By definition, there exists \( S \in \mathbb{Z}_n^{(h)} \) such that \( S - S = \mathbb{Z}_n \). Let \( \mathcal{A} = \{ S + j : j \in \mathbb{Z}_n \} \subset \mathbb{Z}_n^{(h)} \) denote the family of all the translates of \( S \). Then \( \mathcal{A} \) is clearly symmetric and intersecting. Hence, \( g(n) \leq h \), proving the first part of the claim.

Now suppose that \( n \) is prime, and let \( g(n) = k \). Let \( \mathcal{A} \subset [n]^{(k)} \) be a nonempty, symmetric intersecting family. Since \( \text{Aut}(\mathcal{A}) \leq S_n \) is transitive, the orbit-stabilizer theorem implies that \( n \) divides \( \text{Aut}(\mathcal{A}) \), and therefore by Sylow’s theorem, \( \text{Aut}(\mathcal{A}) \) has a cyclic subgroup \( H \) of order \( n \). Let \( \sigma \in S_n \) be a generator of \( H \); then \( \sigma \) is an \( n \)-cycle, and by relabelling the ground set \( [n] \) if necessary, we may assume that \( \sigma = (1 \, 2 \, \ldots \, n) \) (in the standard cycle notation). Fix \( x \in \mathcal{A} \) and note that \( \mathcal{B} = \{ x, \sigma(x), \ldots , \sigma^{n-1}(x) \} \) is also a nonempty, symmetric intersecting family as
Clearly, $\mathcal{B}$ consists of all the cyclic translates, modulo $n$, of $x$. If we regard $x$ as a subset of $\mathbb{Z}_n$, then since $\mathcal{B}$ is intersecting, we have $x - x = \mathbb{Z}_n$, i.e., $x$ is a difference cover for $\mathbb{Z}_n$. Hence, $h(\mathbb{Z}_n) \leq k$ and it follows that $h(\mathbb{Z}_n) = g(n)$ when $n$ is prime, as required.

We now describe how existing constructions of difference covers lead to an improvement of (5). We say that $S \subset \mathbb{Z}$ is a difference cover for $n$ if $[n] \subset S - S$. For each $n \in \mathbb{N}$, let $\pi_n : \mathbb{Z} \to \mathbb{Z}_n$ denote the natural projection modulo $n$ defined by $\pi_n(i) = i \pmod{n}$ for all $i \in \mathbb{Z}$. Note that if $S \subset \mathbb{Z}$ is a difference cover for $\lfloor n/2 \rfloor$, then $\pi_n(S)$ is a difference cover for $\mathbb{Z}_n$. Building on work of Rédei and Rényi [20] and of Leech [19], Golay [15] proved that for any $n \in \mathbb{N}$, there exists a difference cover for $n$ of size at most $\sqrt{cn}$, where $c < 2.6572$ is an absolute constant. It follows that for any $n \in \mathbb{N}$, we have

$$g(n) \leq h(\mathbb{Z}_n) \leq 1.1527\sqrt{n}.$$ 

Unfortunately, one cannot hope to answer Question 4.1 in the affirmative purely by projecting difference covers for $\lfloor n/2 \rfloor$ into $\mathbb{Z}_n$ and using the fact that $g(n) \leq h(\mathbb{Z}_n)$; this is a consequence of a result of Rédei and Rényi [20] which asserts that if $S \subset \mathbb{Z}$ is a difference cover for $n$, then

$$|S| \geq \sqrt{\left(2 + \frac{4}{3\pi}\right)n}.$$ 

In view of Lemma 4.2, we are led to the following question, which being a natural question in its own right, has also occurred independently to others; see [1], for instance.

**Question 4.3.** Is it true that $h(\mathbb{Z}_n) = (1 + o(1))\sqrt{n}$ for all $n \in \mathbb{N}$?

By Lemma 4.2, an affirmative answer to this question would imply an affirmative answer to Question 4.1. We remark that Question 4.3 is a ‘covering’ problem whose ‘packing’ counterpart has received a lot of attention. If $G$ is an Abelian group and $S \subset G$, we say that $S$ is a Sidon set in $G$ if for any non-identity element $g \in G$, there exists at most one ordered pair $(s_1, s_2) \in S^2$ such that $g = s_1 - s_2$. For $n \in \mathbb{N}$, let

$$s(n) = \max\{|S| : S \subset \mathbb{Z}_n \text{ such that } S \text{ is a Sidon set}\}.$$ 

The determination of $s(n)$ is a well-known open problem; see [6], for example. In particular, the following remains open.
Question 4.4. Is it true that \( s(n) = (1 - o(1))\sqrt{n} \) for all \( n \in \mathbb{N} \)?

The constructions of Singer [23] and Bose [3] yield affirmative answers to Question 4.4 when \( n \) is of the form \( q^2 + q + 1 \) or \( q^2 - 1 \) respectively, where \( q \) is a prime power, and a construction due to Ruzsa [21] does so when \( n \) is of the form \( p^2 - p \), where \( p \) is prime.

Returning to the question of determining \( g(n) \), we have shown that

\[
\lfloor \sqrt{n} \rfloor + 1 \leq g(n) \leq 1.1527\sqrt{n}
\]

for all \( n \geq 2 \). It turns out that the precise value of \( g(n) \) has a nontrivial dependence on the arithmetic properties of \( n \). We demonstrate this by showing that the lower bound in (6) is sharp for some positive integers, but strict for others.

Let us first show that the lower bound in (6) is sharp for infinitely many \( n \in \mathbb{N} \). We begin by observing that if \( d \geq 2 \) and there exists a transitive projective plane of order \( d \), then \( s(d^2 + d + 1, k) > 0 \) if and only if \( k \geq d + 1 \). Indeed, if \( k \leq d \), then by Proposition 2.1, \( s(d^2 + d + 1, k) = 0 \). Now, if \( k \geq d + 1 \), then let \( P \) be a transitive projective plane of order \( d \), let \( n = d^2 + d + 1 \), identify \( [n] \) with the set of points of \( P \), and take \( A \) to be the family of all \( k \)-element subsets of the points of \( P \) that contain a line of \( P \). It is clear that \( A \) is nonempty, symmetric and intersecting. If \( k = d + 1 \), then \( |A| = d^2 + d + 1 \), and if \( k > d + 1 \), then writing \( D = d^2 + d + 1 \) and using the Bonferroni inequalities, we have

\[
|A| \geq D \left( \frac{n - d - 1}{k - d - 1} \right) - \left( \frac{D}{2} \right) \left( \frac{n - 2d - 1}{k - 2d - 1} \right) \geq \frac{D}{2} \left( \frac{n - d - 1}{k - d - 1} \right).
\]

(7)

Of course, for any odd prime power \( q \), there exists a transitive projective plane of order \( q \), namely, the Desarguesian projective plane \( \mathbb{P}^2(\mathbb{F}_q) \) over the finite field \( \mathbb{F}_q \). Hence, for any odd prime power \( q \), we have \( s(q^2 + q + 1, k) > 0 \) if and only if \( k \geq q + 1 \); it follows that the lower bound in (6) is sharp for infinitely many positive integers. Moreover, using (7), we have

\[
s(q^2 + q + 1, k) \geq |A| \geq \frac{q^2 + q + 1}{2} \left( \frac{q^2}{k - q - 1} \right)
\]

for all prime powers \( q \) and all \( k \geq q + 1 \). It can be checked from this that for any fixed \( \delta > 0 \) and all sufficiently large prime powers \( q \), we have \( |A| > |\mathcal{F}(q^2 + q + 1, k)| \) for all \( q + 1 \leq k \leq (1 - \delta)q \log q \).
Remark. It is a long-standing conjecture that all transitive finite projective planes are Desarguesian; if this is true, then the above argument would of course apply only for prime powers.

Next, we show that the lower bound in (6) is not tight for $n = 43$. For this, we will need the following lemma, which is perhaps of independent interest.

**Lemma 4.5.** Suppose that $d \geq 2$ and that $d^2 + d + 1$ is prime. If there exists a projective plane of order $d$, then $s(d^2 + d + 1, d + 1) = d^2 + d + 1$; otherwise, $s(d^2 + d + 1, d + 1) = 0$.

**Proof.** We set $n = d^2 + d + 1$ and $k = d + 1$. If there exists a projective plane of order $d$, then $s(n, k) \geq d^2 + d + 1$ as observed above, so we may turn our attention to establishing upper bounds for $s(n, k)$.

Suppose there exists a nonempty, symmetric intersecting family $A \subset [n]^{(k)}$. Since $n$ is prime, the proof of Lemma 4.2 implies that there exists a bijection $\phi: [n] \to \mathbb{Z}_n$ such that for any $x \in A$, we have

1. $\phi(x) - \phi(x) = \mathbb{Z}_n$, and
2. each translate of $\phi(x)$ belongs to $A$.

Furthermore, since $n - 1 = k(k - 1)$, each element of $\mathbb{Z}_n \setminus \{0\}$ has a unique representation of the form $i - j$ with $i, j \in \phi(x)$. It follows that the translates of $\phi(x)$ form the set of lines of a (cyclic) projective plane of order $k - 1 = d$, with $\mathbb{Z}_n$ being its set of points. If there exists no projective plane of order $d$, then we have a contradiction, so $A = \emptyset$, as required. Otherwise, $A$ contains the set of lines of a projective plane of order $d$. It therefore suffices to prove the following well-known claim, whose proof we include for the reader’s convenience.

**Claim 4.6.** Let $d \in \mathbb{N}$ with $d \geq 2$ and let $n = d^2 + d + 1$. If $L$ is the set of lines of a projective plane of order $d$ with point-set $[n]$, then $L$ is a maximal intersecting subfamily of $[n]^{(d+1)}$.

**Proof.** Suppose for the sake of a contradiction that $L$ is not maximal; then there exists $y \in [n]^{(d+1)} \setminus L$ such that $y \cap \ell \neq \emptyset$ for all $\ell \in L$. Let $p_1, p_2 \in y$ be two distinct points in $y$, let $\ell_0$ be the unique line through $p_1$ and $p_2$, and choose $p_3 \in \ell_0 \setminus y$. Note that $p_3$ belongs to exactly $d + 1$ lines; each of these lines intersects $y$, and any two of these lines intersect only at $p_3$ (which is not in $y$). Since one of these $d + 1$...
lines (namely, \( \ell_0 \)) contains at least two points of \( y \) (namely, \( p_1 \) and \( p_2 \)), it follows that \( |y| > d + 2 \), a contradiction. \( \square \)

It now follows that \( \mathcal{A} \) consists precisely of the translates of \( \phi(x) \), and consequently, \( |\mathcal{A}| = n = d^2 + d + 1 \). This completes the proof of the lemma. \( \square \)

It follows from Lemma 4.5 that \( s(43, 7) = 0 \), since 43 is prime and there exists no projective plane of order 6. Hence, the lower bound in (6) is not sharp for \( n = 43 \).

**Remark.** Lemma 4.5 also implies that \( s(d^2 + d + 1, d + 1) = d^2 + d + 1 \) for all \( d \in \{2, 3, 5, 8, 17, 27, 41, 59, 71, 89\} \), since each such \( d \) is a prime power with \( d^2 + d + 1 \) prime. The well-known (and widely believed) Generalized Bunyakovskiy conjecture (due to Schinzel and Sierpiński [22], generalizing simultaneously the conjecture of Bunyakovskiy [4] and the conjecture of Dickson [7]) would imply that \( d^2 + d + 1 \) is prime for infinitely many primes \( d \), which would imply that \( s(d^2 + d + 1, d + 1) = d^2 + d + 1 \) for infinitely many \( d \in \mathbb{N} \); unfortunately, the Generalized Bunyakovskiy conjecture is not currently known to hold in any non-linear case (and in fact, nor is the original Bunyakovskiy conjecture). The even stronger Bateman–Horn conjecture [2] would imply that \( d^2 + d + 1 \) is prime for infinitely many positive integers \( d \) which are not prime powers; together with Lemma 4.5 and the non-existence conjecture for projective planes whose order is not a prime power, this would imply that \( s(d^2 + d + 1, d + 1) = 0 \) for infinitely many \( d \in \mathbb{N} \), and consequently that the lower bound in (6) is not sharp infinitely often. We briefly remind the reader that the Bateman–Horn conjecture implies that for all \( x \geq 0 \), the number of positive integers \( d \leq x \) such that \( d^2 + d + 1 \) is prime is \( \Theta(x/\log x) \); of course, this implies the claim above since, trivially, the number of integers \( d \leq x \) which are of the form \( y^2 \) for some \( y, z \in \mathbb{N} \) with \( z \geq 2 \) is \( \Theta(\sqrt{x}) \).

The rest of this section is devoted to answering Question 4.1 in the affirmative for various sets of positive integers (with suitable ‘arithmetic structure’).

We saw from our projective construction earlier that if \( n = q^2 + q + 1 \) for some prime power \( q \), then \( s(n, k) > 0 \) if and only if \( k \geq q + 1 \); in particular, this implies that \( g(n) = (1 + o(1))\sqrt{n} \). Consequently, we have an affirmative answer to Question 4.1 for any \( n = q^2 + q + 1 \), where \( q \) is a prime power.

Next, we use dual affine planes to answer Question 4.1 affirmatively for another infinite set of positive integers. Let \( q \) be a prime power. Recall that the affine plane \( \mathbb{A}^2(\mathbb{F}_q) \) over \( \mathbb{F}_q \) is the incidence geometry with point-set \( \mathbb{F}_q^2 \) whose lines are
the 1-dimensional affine subspaces of $\mathbb{P}^2_q$: in other words, the lines of $A^2(\mathbb{F}_q)$ are the $q^2 + q$ sets of the form $\{x + \lambda v : \lambda \in \mathbb{F}_q\}$, where $x \in \mathbb{F}_q^2$ and $v \in \mathbb{F}_q^2 \setminus \{0\}$, so it is clear that each point lies on $q + 1$ lines. The dual affine plane $\mathbb{DA}^2(\mathbb{F}_q)$ over $\mathbb{F}_q$ is obtained by interchanging the point-set and the line-set of $A^2(\mathbb{F}_q)$ and preserving the incidence relation, so $\mathbb{DA}^2(\mathbb{F}_q)$ has $q^2 + q$ points and $q^2$ lines, and each line contains $q + 1$ points. If $n = q^2 + q$ and $k = q + 1$, then we identify $[n]$ with the point-set of $\mathbb{DA}^2(\mathbb{F}_q)$, and take $\mathcal{A} \subset [n]^{(k)}$ to be the family of lines of $\mathbb{DA}^2(\mathbb{F}_q)$. We claim that $\mathcal{A}$ is a symmetric intersecting family. Indeed, any two points in $A^2(\mathbb{F}_q)$ lie on a common line in $A^2(\mathbb{F}_q)$, so $\mathcal{A}$ is intersecting. For any two lines $\ell_1, \ell_2$ in $A^2(\mathbb{F}_q)$, there is an affine transformation $\sigma \in \text{Aff}(\mathbb{F}_q^2)$, i.e., a map of the form $v \mapsto Mv + c$ for some $M \in \text{GL}(\mathbb{F}_q^2)$ and $c \in \mathbb{F}_q^2$, such that $\sigma(\ell_1) = \ell_2$; clearly, $\sigma$ defines an automorphism of $\mathcal{A}$, so $\mathcal{A}$ is symmetric. It follows that

$$s(q^2 + q, q + 1) \geq |\mathcal{A}| = q^2 > 0.$$ 

In conjunction with Proposition 2.1, this implies that for any odd prime power $q$, we have $s(q^2 + q, k) > 0$ if and only if $k \geq q + 1$. Therefore, we have an affirmative answer to Question 4.1 for any $n = q^2 + q$, where $q$ is a prime power; indeed, the lower bound in (6) is sharp for all $n$ of this form.

Our constructions of symmetric intersecting families based on projective planes and dual affine planes have natural analogues based upon higher-dimensional projective spaces and higher-dimensional dual affine spaces, enabling us to answer Question 4.1 affirmatively for some other infinite sets of integers.

We first describe a construction based upon higher-dimensional projective spaces. Fix $r \in \mathbb{N}$ and let $q$ be a prime power. If $n = (q^{2r+1} - 1)/(q - 1)$ and $k = (q^{r+1} - 1)/(q - 1)$, then we identify $[n]$ with the set of points of the $(2r)$-dimensional projective space $\mathbb{P}^{2r}(\mathbb{F}_q)$, and take $\mathcal{A}$ to be the family of all $r$-dimensional projective subspaces. Clearly, $\mathcal{A}$ is symmetric and intersecting, so

$$s\left(\frac{q^{2r+1} - 1}{q - 1}, \frac{q^{r+1} - 1}{q - 1}\right) \geq \binom{2r + 1}{r + 1} q^r \prod_{i=0}^{r} \frac{q^{2r+1-i} - 1}{q^{i+1} - 1} > 0.$$

Note that if $r \in \mathbb{N}$ is fixed, then $k = (1 + o(1))\sqrt{n}$ as $q \to \infty$; this gives an affirmative answer to Question 4.1 for all $n$ of the form above.

We next describe a construction based upon higher-dimensional dual affine spaces. Again, fix $r \in \mathbb{N}$ and let $q$ be a prime power. Let $A^{2r}(\mathbb{F}_q)$ denote the $(2r)$-dimensional affine space over $\mathbb{F}_q$ so that for each $i \in [2r - 1] \cup \{0\}$, the $i$-flats
of $\mathbb{A}^{2r}(\mathbb{F}_q)$ are the $i$-dimensional affine subspaces of $\mathbb{F}_q^{2r}$; in particular, the 0-flats are the points and the $(2r - 1)$-flats are the affine hyperplanes, so there are $q^{2r}$ points and $q(q^{2r} - 1)/(q - 1)$ affine hyperplanes. Two flats are said to be incident if one is contained in the other. It is easy to see that a fixed $(r - 1)$-flat of $\mathbb{A}^{2r}(\mathbb{F}_q)$ is contained in $(q^{r+1} - 1)/(q - 1)$ hyperplanes, and that there are

\[
q^{r+1} \begin{bmatrix} 2r \\ r - 1 \end{bmatrix}_q = q^{r+1} \prod_{i=0}^{r-2} q^{2r-i-1} - 1 
\]

$(r - 1)$-flats. The $(2r)$-dimensional dual affine space $\mathbb{D}\mathbb{A}^{2r}(\mathbb{F}_q)$ is the space whose $i$-flats are the $(2r - i - 1)$-flats of $\mathbb{A}^{2r}(\mathbb{F}_q)$ for each $i \in [2r - 1] \cup \{0\}$. In particular, the points of $\mathbb{D}\mathbb{A}^{2r}(\mathbb{F}_q)$ are the affine hyperplanes of $\mathbb{A}^{2r}(\mathbb{F}_q)$ and the $r$-flats of $\mathbb{D}\mathbb{A}^{2r}(\mathbb{F}_q)$ are the $(r - 1)$-flats of $\mathbb{A}^{2r}(\mathbb{F}_q)$ and again, two flats are incident if one is contained in the other. Now, if $n = q(q^{2r} - 1)/(q - 1)$ and $k = (q^{r+1} - 1)/(q - 1)$, then we identify $[n]$ with the point-set of $\mathbb{D}\mathbb{A}^{2r}(\mathbb{F}_q)$, and take $\mathcal{A} \subset [n]^{(k)}$ to be the family of all $r$-flats of $\mathbb{D}\mathbb{A}^{2r}(\mathbb{F}_q)$. We claim that $\mathcal{A}$ is a symmetric intersecting family. Indeed, any two $(r - 1)$-flats of $\mathbb{A}^{2r}(\mathbb{F}_q)$ are contained in a common affine hyperplane in $\mathbb{A}^{2r}(\mathbb{F}_q)$, so $\mathcal{A}$ is intersecting. Also, for any two affine hyperplanes $V_1$ and $V_2$ in $\mathbb{A}^{2r}(\mathbb{F}_q)$, there exists $\sigma \in \text{Aff}(\mathbb{F}_q^{2r})$, i.e., a map of the form $v \mapsto Mv + c$, where $M \in \text{GL}(\mathbb{F}_q^{2r})$ and $c \in \mathbb{F}_q^{2r}$, such that $\sigma(V_1) = V_2$; clearly, $\sigma$ defines an automorphism of $\mathcal{A}$, so $\mathcal{A}$ is symmetric. Hence,

\[
s \left( \frac{q(q^{2r} - 1)}{q - 1}, \frac{q^{r+1} - 1}{q - 1} \right) \geq q^{r+1} \begin{bmatrix} 2r \\ r - 1 \end{bmatrix}_q = q^{r+1} \prod_{i=0}^{r-2} q^{2r-i-1} - 1 > 0.
\]

Note as before that if $r \in \mathbb{N}$ is fixed, then $k = (1 + o(1))\sqrt{n}$ as $q \to \infty$; this gives an affirmative answer to Question 4.1 for all $n$ of the form above.

Finally, we demonstrate using a tensor product construction that $\mathcal{S}$ is closed under taking pointwise products. For a set $x \subset [n]$, we define its characteristic vector $\chi_x \in \{0, 1\}^n$ by $(\chi_x)_i = 1$ if $i \in x$ and $(\chi_x)_i = 0$ otherwise. Given two sets $x \subset [n]$ and $y \subset [m]$, we define their tensor product $x \otimes y$ to be the subset of $[nm]$ whose characteristic vector $\chi_{x \otimes y}$ is given by

\[
(\chi_{x \otimes y})_{(i-1)m+j} = (\chi_x)_i(\chi_y)_j
\]

for all $i \in [n]$ and $j \in [m]$. For two families $\mathcal{A} \subset \mathcal{P}_n$ and $\mathcal{B} \subset \mathcal{P}_m$, we define their tensor product by

\[
\mathcal{A} \otimes \mathcal{B} = \{ x \otimes y : x \in \mathcal{A}, y \in \mathcal{B} \};
\]
note that $\mathcal{A} \otimes \mathcal{B} \subset \mathcal{P}_{nm}$ and that $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}||\mathcal{B}|$. Now observe that if $\mathcal{A} \subset [n]\choose{k}$ and $\mathcal{B} \subset [m]\choose{l}$, then $\mathcal{A} \otimes \mathcal{B} \subset nm\choose{kl}$, and furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are symmetric and intersecting, then so is $\mathcal{A} \otimes \mathcal{B}$. It follows that

$$s(nm, kl) \geq s(n, k)s(m, l)$$

for all $k, l, m, n \in \mathbb{N}$, and in particular, if $(n, k), (m, l) \in \mathcal{S}$, then $(nm, kl) \in \mathcal{S}$. Therefore, the function $g$ is submultiplicative, i.e., we have

$$g(nm) \leq g(n)g(m)$$

for all $n, m \in \mathbb{N}$. The fact that $\mathcal{S}$ is closed under pointwise products may be used to answer Question 4.1 affirmatively for some additional sets of positive integers. For example, if $q_1, \ldots, q_r$ are prime powers, then

$$s\left(\prod_{i=1}^{r}(q_i^2 + q_i + 1), \prod_{i=1}^{r} q_i\right) \geq \prod_{i=1}^{r}(q_i^2 + q_i + 1) > 0,$$

so if $n = \prod_{i=1}^{r}(q_i^2 + q_i + 1)$ and $k = \prod_{i=1}^{r} q_i$, then $s(n, k) > 0$, and if we additionally have $\sum_{i=1}^{r} q_i^{-1} = o(1)$, then $k = (1+o(1))\sqrt{n}$; this answers Question 4.1 affirmatively for all $n$ of this form.

5. Conclusion

A number of open problems remain. Theorem 1.3 and Lemma 3.2 together determine the order of magnitude of $\log((n)\choose{k}/s(n, k))$ when $k/n$ is bounded away from zero by a positive constant. The gap between our upper and lower bounds for $s(n, k)$ is somewhat worse for smaller $k$, and it would be of interest to improve Theorem 1.3 in the regime where $k = o(n)$.

Determining $s(n, k)$ precisely for all $k \leq n/2$ would appear to be a challenging problem. We conjecture that for any $\delta > 0$, if $n$ is sufficiently large depending on $\delta$ and $(1 + \delta)\sqrt{n}\log n \leq k \leq n/2$, then

$$s(n, k) = |\mathcal{F}(n, k)|.$$

Note that if $n$ is sufficiently large depending on $\delta$, then the family $\mathcal{F}(n, k)$ yields a larger symmetric intersecting family than any of the algebraic constructions in Section 4 provided $(1 + \delta)\sqrt{n}\log n \leq k \leq n/2$.

Determining the asymptotic behaviour of $g(n)$ is another problem that merits further investigation. We have established various estimates in Section 4, but the
basic question of deciding whether $g(n)/\sqrt{n}$ converges in the limit as $n \to \infty$ still remains open.

Let us mention one other problem similar in flavour to those considered above. In [5], Cameron, Frankl and Kantor raise the following question: for which $n \in \mathbb{N}$ does there exist a symmetric intersecting family $A \subset \mathcal{P}_n$ with $|A| = 2^{n-1}$? They obtain several results on this problem, but a complete answer is not yet known.

Finally, we remark that the proof of Theorem 1.3 has some ingredients in common with the recent proof of a conjecture of Frankl (on symmetric 3-wise intersecting subfamilies of $\mathcal{P}_n$) by the first and last authors [10], namely, the consideration of the function $p \mapsto \mu_p(\mathcal{F})$ for a suitable symmetric increasing family $\mathcal{F} \subset \mathcal{P}_n$, and the application of a sharp threshold result to this function. We believe that it is somewhat more surprising that this technique should be useful in the setting of [10] than in the current paper. Indeed, in the setting of [10], it is not immediately obvious that the $p$-biased measure (with $p \neq 1/2$) will be of any use in analysing the size of families with respect to the uniform measure on $\mathcal{P}_n$. By contrast, in the current setting, the strategy of approximating the uniform measure on $[n](k)$ in terms of the $p$-biased measure on $\mathcal{P}_n$, where $p \approx k/n$, is known to be useful in the study of intersecting families, for example from the work of Friedgut [13], and of Dinur and Friedgut [8], as mentioned in Section 2.

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