The purpose of this page is to spell out in detail the example given in lectures on applying Theorem 4.8 (Lindelöf–Picard) to n^{th}-order ODEs.

Example. Let $a < b$ and $R > 0$ be real numbers, let $z = (z_0, z_1, \dots, z_{n-1}) \in \mathbb{R}^n$ and let

$$\psi\colon [a,b] \times B_R(z) \to \mathbb{R}$$

be a continuous function. Assume that for some $K > 0$ we have

$$|\psi(t,x) - \psi(t,y)| \leqslant K\|x-y\| \qquad \text{for all } t \in [a,b] \text{ and all } x,y \in B_R(z)\ .$$

Then there exists $\varepsilon > 0$ such the for any $t_0 \in [a,b]$ the n^{th}-order IVP (initial value problem)

$$\begin{aligned} g^{(n)}(t) &= \psi(t, g(t), g^{(1)}(t), g^{(2)}(t), \dots, g^{(n-1)}(t)) \\ g^{(j)}(t_0) &= z_j \quad \text{for } 0 \leqslant j \leqslant n-1 \end{aligned} \tag{1}$$

has a unique solution on $[c,d] = [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a,b]$.

Note. This means that there is a unique n-times differentiable function

$$g\colon [c,d] \to \mathbb{R}$$

that satisfies (1) for all $t \in [c,d]$. This implicitly includes the assumption that

$$(g(t), g^{(1)}(t), g^{(2)}(t), \dots, g^{(n-1)}(t)) \in B_R(z)$$

for all $t \in [c,d]$.

Proof. Let us define $\varphi\colon [a,b] \times B_R(z) \to \mathbb{R}^n$ by setting

$$\varphi(t, x_0, x_1, \dots, x_{n-1}) = (x_1, \dots, x_{n-1}, \psi(t, x_0, x_1, \dots, x_{n-1}))$$

for $t \in [a,b]$ and $x = (x_0, x_1, \dots, x_{n-1}) \in B_R(z)$. Then φ is continuous and satisfies

$$\|\varphi(t,x) - \varphi(t,y)\| \leqslant (K+1)\|x-y\| \qquad \text{for all } t \in [a,b] \text{ and all } x,y \in B_R(z)\ .$$

By Lindelöf–Picard (Theorem 4.8 in the lectures), there exists $\varepsilon > 0$ such that the IVP

$$f'(t) = \varphi(t, f(t))\ , \qquad f(t_0) = z \tag{2}$$

has a unique solution on $[c,d] = [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a,b]$. Let f be this unique solution. Thus, $f\colon [c,d] \to B_R(z)$ is a differentiable function with $f(t_0) = z$ and $f'(t) = \varphi(t, f(t))$ for all $t \in [c,d]$. Let $f_0, f_1, \dots, f_{n-1}$ be the components of f, *i.e.*, functions $f_j\colon [c,d] \to \mathbb{R}$ such that $f(t) = (f_0(t), f_1(t), \dots, f_{n-1}(t))$ for all $t \in [c,d]$. Since f is a solution of (2), each f_j is differentiable and

$$\begin{aligned} (f_0'(t), f_1'(t), \dots, f_{n-1}'(t)) &= f'(t) = \varphi(t, f(t)) \\ &= (f_1(t), f_2(t), \dots, f_{n-1}(t), \psi(t, f_0(t), f_1(t), \dots, f_{n-1}(t))) \end{aligned} \tag{3}$$

for all $t \in [c,d]$. Set $g = f_0$. Comparing coordinates in (3) shows that g is an n-times differentiable function $[c,d] \to \mathbb{R}$ with $g^{(j)} = f_j$ for $0 \leqslant j < n$ (induction on j), and moreover

$$\begin{aligned} g^{(n)}(t) = f_{n-1}'(t) &= \psi(t, f_0(t), f_1(t), \dots, f_{n-1}(t)) \\ &= \psi(t, g(t), g^{(1)}(t), \dots, g^{(n-1)}(t)) \end{aligned}$$

for all $t \in [c,d]$. Finally, since $f(t_0) = z$, we have $g^{(j)}(t_0) = f_j(t_0) = z_j$ for $0 \leqslant j \leqslant n-1$. This completes the proof of existence.

To prove uniqueness, assume that $\tilde{g}$ is another solution to (1) on $[c,d]$. Define $\tilde{f}\colon [c,d] \to B_R(z)$ by setting $\tilde{f}(t) = (\tilde{g}(t), \tilde{g}^{(1)}(t), \dots, \tilde{g}^{(n-1)}(t))$. It is straightforward to verify that $\tilde{f}$ is a solution to (2). It follows that $\tilde{f} = f$ and $\tilde{g} = g$. $\square$