Part III Functional Analysis

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1 The Hahn–Banach extension theorems

Dual space. Let X be a normed space. The *dual space* X^* *of* X is the space of all bounded linear functionals on X. The dual space is a Banach space in the operator norm which is defined for $f \in X^*$ as

$$\|f\| = \sup\big\{|f(x)| : x \in B_X\big\} .$$

Recall that $B_X = \{x \in X : \|x\| \leqslant 1\}$ is the closed unit ball of X and that $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of X.

Examples. $\ell_p^* \cong \ell_q$ for $1 \leqslant p < \infty$, $1 < q \leqslant \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We also have $\mathrm{c}_0^* \cong \ell_1$. If H is a Hilbert space then $H^* \cong H$ (conjugate-linear in the complex case).

Notation. 1. Given normed spaces X and Y, we write $X \sim Y$ if X and Y are isomorphic, *i.e.*, when there is a linear bijection $T\colon X \to Y$ such that both T and T^{-1} are continuous. (Recall that by the Open Mapping Theorem, if X and Y are both complete and T is a continuous linear bijection, then T^{-1} is automatically continuous.)

2. Given normed spaces X and Y, we write $X \cong Y$ if X and Y are isometrically isomorphic, *i.e.*, when there is a surjective linear map $T\colon X \to Y$ such that $\|Tx\| = \|x\|$ for all $x \in X$. It follows that T is a continuous linear bijection and that T^{-1} is also isometric, and hence continuous.
3. For $x \in X$ and $f \in X^*$, we shall sometimes denote $f(x)$, the action of f on x, by $\langle x, f\rangle$. By definition of the operator norm we have

$$|\langle x, f\rangle| = |f(x)| \leqslant \|f\| \cdot \|x\| \ .$$

When X is a Hilbert space and we identify X^* with X in the usual way, then $\langle x, f\rangle$ is the inner product of x and f.

Definition. Let X be a real vector space. A functional $p\colon X \to \mathbb{R}$ is called

- *positive homogeneous* if $p(tx) = tp(x)$ for all $x \in X$ and $t \geqslant 0$, and
- *subadditive* if $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$.

Theorem 1. (Hahn–Banach) Let X be a real vector space and p be a positive homogeneous, subadditive functional on X. Let Y be a subspace of X and $g\colon Y \to \mathbb{R}$ be a linear functional such that $g(y) \leqslant p(y)$ for all $y \in Y$. Then there is a linear functional $f\colon X \to \mathbb{R}$ such that $f\restriction_Y = g$ and $f(x) \leqslant p(x)$ for all $x \in X$.

Zorn's Lemma. This is used in the proof, so we recall it here. Let $(P, \leqslant)$ be a poset (partially ordered set). An element $x \in P$ is an *upper bound* for a subset A of P if $a \leqslant x$ for all $a \in A$. A subset C of P is called a *chain* if it is linearly ordered by $\leqslant$. An element $x \in P$ is a *maximal element* of P if whenever $x \leqslant y$ for some $y \in P$, we have $y = x$. Zorn's Lemma states that if $P \neq \emptyset$ and every non-empty chain in P has an upper bound, then P has a maximal element.

Definition. A *seminorm* on a real or complex vector space X is a functional $p\colon X \to \mathbb{R}$ such that

- $p(x) \geqslant 0$ for all $x \in X$;
- $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and for all scalars λ;
- $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$.

Note. "norm" $\Longrightarrow$ "seminorm" $\Longrightarrow$ "positive homogeneous+subadditive".

Theorem 2. (Hahn–Banach) Let X be a real or complex vector space and p be a seminorm on X. Let Y be a subspace of X and g be a linear functional on Y such that $|g(y)| \leqslant p(y)$ for all $y \in Y$. Then there is a linear functional f on X such that $f\restriction_Y = g$ and $|f(x)| \leqslant p(x)$ for all $x \in X$.

Remark. It follows from the proof given in lectures that for a complex normed space X, the map $f \mapsto \mathrm{Re}(f)\colon \left(X^*\right)_{\mathbb{R}} \to \left(X_{\mathbb{R}}\right)^*$ is an isometric isomorphism. Here $Y_{\mathbb{R}}$, for a complex vector space Y, denotes the real vector space obtained from Y by restricting scalar multiplication to the reals.

Corollary 3. Let X be a real or complex vector space and p be a seminorm on X. For every $x_0 \in X$ there is a linear functional f on X with $f(x_0) = p(x_0)$ and $|f(x)| \leqslant p(x)$ for all $x \in X$.

Theorem 4. (Hahn–Banach) Let X be a real or complex normed space. Then

(i) Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ with $f\restriction_Y = g$ and $\|f\| = \|g\|$.

(ii) Given a non-zero $x_0 \in X$, there exists $f \in S_{X^*}$ with $f(x_0) = \|x_0\|$.

Remarks. 1. Part (i) can be viewed as a linear version of Tietze's extension theorem. The latter states that if L is a closed subset of a compact Hausdorff space K, and $g\colon L \to \mathbb{R}$ (or $\mathbb{C}$) is continuous, then there is a continuous function $f\colon K \to \mathbb{R}$ (respectively, $\mathbb{C}$) such that $f\restriction_L = g$ and $\|f\|_\infty = \|g\|_\infty$.
2. Part (ii) implies that X^* separates the points of X: given $x \neq y$ in X, there exists $f \in X^*$ such that $f(x) \neq f(y)$. Thus there are "plenty" bounded linear functionals on X.
3. The element $f \in X^*$ in part (ii) is called a *norming functional for* x_0. It shows that

$$\|x_0\| = \max\left\{|\langle x_0, g\rangle| : g \in B_{X^*}\right\} .$$

Another name for f is *support functional at* x_0. Assume X is a real normed space and that $\|x_0\| = 1$. Then $B_X \subset \{x \in X : f(x) \leqslant 1\}$, and hence the hyperplane $\{x \in X : f(x) = 1\}$ can be thought of as a tangent to B_X at x_0.

Bidual. Let X be a normed space. Then $X^{**} = \left(X^*\right)^*$ is called the *bidual* or *second dual of* X. It is the Banach space of all bounded linear functionals on X^* with the operator norm. For $x \in X$ we define $\hat{x}\colon X^* \to \mathbb{R}$ (or $\mathbb{C}$) by $\hat{x}(f) = f(x)$ (*evaluation at* x). Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leqslant \|x\|$. The map $x \mapsto \hat{x}\colon X \to X^{**}$ is called the *canonical embedding*.

Theorem 5. The canonical embedding defined above is an isometric isomorphism of X *into* X^{**}.

Remarks. 1. Using the bracket notation, we have

$$\langle f, \hat{x}\rangle = \langle x, f\rangle = f(x) \qquad x \in X,\ f \in X^* .$$

2. The image $\widehat{X} = \{\hat{x} : x \in X\}$ of the canonical embedding is closed in X^{**} if and only if X is complete.
3. In general, the closure of $\widehat{X}$ in X^{**} is a Banach space which contains an isometric copy of X as a dense subspace. We have thus proved that every normed space has a completion.

Definition. X is *reflexive* if the canonical embedding of X into X^{**} is surjective.

Examples. 1. The spaces ℓ_p, $1 < p < \infty$, Hilbert spaces, finite-dimensional spaces are reflexive. Later we prove that the spaces $L_p(\mu)$, $1 < p < \infty$, are also reflexive.
2. The spaces c_0, ℓ_1, $L_1[0,1]$ are not reflexive.

Remark. Every reflexive space X is complete with $X \cong X^{**}$. There are Banach spaces X with $X \cong X^{**}$ which are not reflexive. So for $1 < p < \infty$, it is not sufficient to say that $\ell_p^{**} \cong \ell_q^* \cong \ell_p$ (where q is the conjugate index of p) to deduce that ℓ_p is reflexive. One also has to verify that this isometric isomorphism is the canonical embedding.

Dual operators. Recall that for normed spaces X, Y we denote by $\mathcal{B}(X, Y)$ the space of all bounded linear maps $T\colon X \to Y$. This is a normed space in the operator norm:

$$\|T\| = \sup\left\{\|Tx\| : x \in B_X\right\} .$$

Moreover, $\mathcal{B}(X, Y)$ is complete if Y is.

For $T \in \mathcal{B}(X, Y)$, the *dual operator* of T is the map $T^*\colon Y^* \to X^*$ defined by $T^*(g) = g \circ T$ for $g \in Y^*$. In bracket notation:

$$\langle x, T^* g\rangle = \langle Tx, g\rangle \qquad x \in X,\ g \in Y^* .$$

T^* is linear and bounded with $\|T^*\| = \|T\|$ (uses Theorem 4(ii)).

Remark. When X and Y are Hilbert spaces, the dual operator T^* corresponds to the adjoint of T after identifying X^* and Y^* with X and Y, respectively.

Example. Let $1 < p < \infty$ and consider the right shift $R\colon \ell_p \to \ell_p$. Then $R^*\colon \ell_q \to \ell_q$ (q the conjugate index of p) is the left shift.

Properties of dual operators. 1. $\left(I_X\right)^* = I_{X^*}$
2. $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ for scalars λ, μ and for $S, T \in \mathcal{B}(X, Y)$. Note that unlike for adjoints there is no complex conjugation here. That is because the identification of a Hilbert space with its dual is conjugate-linear in the complex case.
3. $(ST)^* = T^* S^*$ for $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$.
4. The map $T \mapsto T^*\colon \mathcal{B}(X, Y) \to \mathcal{B}(Y^*, X^*)$ is an *into* isometric isomorphism.
5. The following diagram

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ \downarrow & & \downarrow \\ X^{**} & \xrightarrow{\ T^{**}\ } & Y^{**} \end{array}$$

commutes, where the vertical arrows represent canonical embeddings.

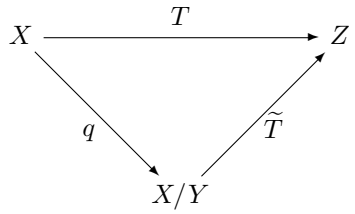
Remark. It follows from the above properties that if $X \sim Y$ then $X^* \sim Y^*$.

Quotient space. Let X be a normed space and Y a closed subspace. Then the quotient space X/Y becomes a normed space in the *quotient norm* defined as follows.

$$\|x + Y\| = \inf\left\{\|x + y\| : y \in Y\right\} \qquad x \in X .$$

The quotient map $q\colon X \to X/Y$ is a surjective linear map that is bounded with $\|q\| \leqslant 1$. It maps the open unit ball $D_X = \{x \in X : \|x\| < 1\}$ onto $D_{X/Y}$. It follows that $\|q\| = 1$ (if $X \neq Y$) and that q is an open map (it maps open sets onto open sets).

If $T\colon X \to Z$ is a bounded linear map with $Y \subset \ker(T)$, then there is a unique map $\widetilde{T}\colon X/Y \to Z$ such that the diagram



commutes, and moreover $\widetilde{T}$ is linear, bounded and $\|\widetilde{T}\| = \|T\|$.

Theorem 6. Let X be a normed space. If X^* is separable, then so is X.

Remark. The converse is false. *E.g.,* $X = \ell_1$ is separable but $X^* \cong \ell_\infty$ is not.

Theorem 7. Every separable normed space is isometrically isomorphic to a subspace of ℓ_∞.

Remarks. 1. The result says that ℓ_∞ is *isometrically universal* for the class $\mathcal{SB}$ of all separable Banach spaces. We will later see that there is a separable space with the same property.
2. A dual version of the above result states that every separable Banach spaces is a quotient of ℓ_1.

Theorem 8. (Vector-valued Liouville) Let X be a complex Banach space and $f\colon \mathbb{C} \to X$ be a bounded, holomorphic function. Then f is constant.

Locally Convex Spaces. A *locally convex space (LCS)* is a pair $(X, \mathcal{P})$, where X is a real or complex vector space and $\mathcal{P}$ is a family of seminorms on X that separates the points of X in the sense that for every $x \in X$ with $x \neq 0$, there is a seminorm $p \in \mathcal{P}$ with $p(x) \neq 0$. The family $\mathcal{P}$ defines a topology on X: a set $U \subset X$ is *open* if and only if for all $x \in U$ there exist $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon > 0$ such that

$$\big\{y \in X : p_k(y - x) < \varepsilon \text{ for } 1 \leqslant k \leqslant n\big\} \subset U \ .$$

Remarks. 1. Addition and scalar multiplication are continuous.
2. The topology of X is Hausdorff as $\mathcal{P}$ separates the points of X.
3. A sequence $x_n \to x$ in X if and only if $p(x_n - x) \to 0$ for all $p \in \mathcal{P}$. (The same holds for nets.)
4. For a subspace Y of X define $\mathcal{P}_Y = \{p\restriction_Y\colon p \in \mathcal{P}\}$. Then $(Y, \mathcal{P}_Y)$ is a locally convex space, and the corresponding locally convex topology on Y is nothing else but the subspace topology on Y induced by X.
5. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of seminorms on X both of which separate the points of X. We say $\mathcal{P}$ and $\mathcal{Q}$ are *equivalent* and write $\mathcal{P} \sim \mathcal{Q}$ if they define the same topology on X.

The topology of a locally convex space $(X, \mathcal{P})$ is metrizable if and only if there is a countable $\mathcal{Q}$ with $\mathcal{Q} \sim \mathcal{P}$.

Definition. A *Fréchet space* is a complete metrizable locally convex space.

Examples. 1. Every normed space $(X, \|\cdot\|)$ is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.
2. Let U be a non-empty, open subset of $\mathbb{C}$, and let $\mathcal{O}(U)$ denote the space of all holomorphic functions $f\colon U \to \mathbb{C}$. For a compact set $K \subset U$ and for $f \in \mathcal{O}(U)$ set $p_K(f) = \sup_{z\in K}|f(z)|$. Set $\mathcal{P} = \{p_K : K \subset U, K \text{ compact}\}$. Then $(\mathcal{O}(U), \mathcal{P})$ is a locally convex space whose topology is the topology of local uniform convergence.

There exist compact sets $K_n \subset U$, $n \in \mathbb{N}$, such that $K_n \subset \operatorname{int}(K_{n+1})$ and $U = \bigcup_n K_n$. Then $\{p_{K_n} : n \in \mathbb{N}\}$ is countable and equivalent to $\mathcal{P}$. Hence $(\mathcal{O}(U), \mathcal{P})$ is metrizable and in fact it is a Fréchet space.

The topology of local uniform convergence is not *normable*: the topology is not induced by a norm. This follows, for example, from Montel's theorem: given a sequence (f_n) in $\mathcal{O}(U)$ such that $(f_n\restriction_K)$ is bounded in $(C(K), \|\cdot\|_\infty)$ for every compact $K \subset U$, there is a subsequence of (f_n) that converges locally uniformly.
3. Fix $d \in \mathbb{N}$ and a non-empty open set $\Omega \subset \mathbb{R}^d$. Let $C^\infty(\Omega)$ denote the space of all infinitely differentiable functions $f\colon \Omega \to \mathbb{R}$. Every *multi-index* $\alpha = (\alpha_1, \dots, \alpha_d) \in \left(\mathbb{Z}_{\geqslant 0}\right)^d$ gives rise to a partial differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} .$$

For a multi-index α, a compact set $K \subset \Omega$, and $f \in \mathcal{O}(U)$ define

$$p_{K,\alpha}(f) = \sup\left\{|(D^\alpha f)(x)| : x \in K\right\} .$$

Set $\mathcal{P} = \left\{p_{K,\alpha} : K \subset \Omega \text{ compact}, \alpha \in (\mathbb{Z}_{\geqslant 0})^d\right\}$. Then $(C^\infty(\Omega), \mathcal{P})$ is a locally convex space. It is a Fréchet space and is not normable.

Lemma 9. Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be locally convex spaces. Let $T\colon X \to Y$ be a linear map. Then TFAE:

(i) T is continuous.

(ii) T is continuous at 0.

(iii) For all $q \in Q$ there exist $n \in \mathbb{N}$, $p_1, \dots, p_n \in \mathcal{P}$ and $C \geqslant 0$ such that

$$q(Tx) \leqslant C \max_{1\leqslant k\leqslant n} p_k(x) \qquad \text{for all } x \in X .$$

Dual space. Let $(X, \mathcal{P})$ be a locally convex space. The *dual space* X^* of X is the space of all linear functionals on X that are continuous with respect to the locally convex topology of X.

Lemma 10. Let f be a linear functional on a locally convex space X. Then $f \in X^*$ if and only if $\ker f$ is closed.

Theorem 11. (Hahn–Banach) Let $(X, \mathcal{P})$ be a locally convex space.

(i) Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ with $f\restriction_Y = g$.

(ii) Given a closed subspace Y of X and $x_0 \in X \setminus Y$, there exists $f \in X^*$ such that $f\restriction_Y = 0$ and $f(x_0) \neq 0$.

Remark. It follows that X^* separates the points of X.

The dual space of $L_p(\mu)$. We fix a measure space $(\Omega, \mathcal{F}, \mu)$. Let $1 \leqslant p < \infty$ and let q be the conjugate index of p (*i.e.*, $1 < q \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$). For $g \in L_q = L_q(\mu)$ define

$$\varphi_g \colon L_p \to \text{scalars} , \qquad \varphi_g(f) = \int_\Omega fg \, \mathrm{d}\mu .$$

By Hölder's inequality we have φ_g is well-defined and $|\varphi_g(f)| \leqslant \|f\|_p \cdot \|g\|_q$. By linearity of integration, φ_g is linear, and hence $\varphi_g \in L_p^*$ with $\|\varphi_g\| \leqslant \|g\|_q$. We have thus obtained a function

$$\varphi \colon L_q \to L_p^* , \quad g \mapsto \varphi_g .$$

This is linear by linearity of integration and bounded with $\|\varphi\| \leqslant 1$.

Theorem 12. Let $(\Omega, \mathcal{F}, \mu)$, p, q, φ be as above.

(i) If $1 < p < \infty$, then φ is an isometric isomorphism. Thus $L_p^* \cong L_q$.

(ii) If $p = 1$ and in addition μ is σ-finite, then φ is an isometric isomorphism. Thus, in this case, we have $L_1^* \cong L_\infty$.

Remarks. 1. Recall that μ is *σ-finite* if there is a sequence (A_n) in $\mathcal{F}$ such that $\Omega = \bigcup A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.
2. One approach for proving surjectivity of φ is via the Radon–Nikodym theorem. We shall follow a different path via uniform convexity in Chapter 4.

***C*(*K*) spaces.** We fix a compact Hausdorff space K. We shall be interested in the following spaces and sets.

$C(K) = \{f \colon K \to \mathbb{C} : \ f \text{ is continuous}\}$

$C^{\mathbb{R}}(K) = \{f \colon K \to \mathbb{R} : \ f \text{ is continuous}\}$

$C^+(K) = \{f \in C(K) : \ f \geqslant 0 \text{ on } K\}$

$M(K) = C(K)^* = \{\varphi \colon C(K) \to \mathbb{C} : \ \varphi \text{ is linear and continuous}\}$

$M^{\mathbb{R}}(K) = \{\varphi \in M(K) : \ \varphi(f) \in \mathbb{R} \text{ for all } f \in C^{\mathbb{R}}(K)\}$

$M^+(K) = \{\varphi \colon C(K) \to \mathbb{C} : \ \varphi \text{ is linear and } \varphi(f) \geqslant 0 \text{ for all } f \in C^+(K)\}$

Note. 1. $C(K)$ is a complex Banach space in the sup-norm: $\|f\|_\infty = \sup_K |f|$.

2. $C^{\mathbb{R}}(K)$ is a real Banach space in the sup-norm.

3. $C^+(K)$ is a subset of $C(K)$. By "$f \geqslant 0$ on K", we of course mean that $f(x) \geqslant 0$ for all $x \in K$.

4. $M^{\mathbb{R}}(K)$ is a closed, real-linear subspace of $M(K)$. $C^{\mathbb{R}}(K)^* \cong M^{\mathbb{R}}(K)$. $f \mapsto f\restriction_{C^{\mathbb{R}}(K)} \colon M^{\mathbb{R}}(K) \to C^{\mathbb{R}}(K)^*$ is an isometric real-linear isomorphism.

5. Elements of $M^+(K)$ are called *positive linear functionals*. These are automatically continuous, and in fact

$$M^+(K) = \{\varphi \in M(K) : \ \|\varphi\| = \varphi(\mathbf{1}_K)\} .$$

The dual space of $C(K)$. Let ν be a complex Borel measure on K, *i.e.*, a countably additive set-function $\nu\colon \mathcal{B} \to \mathbb{C}$ where $\mathcal{B}$ is the Borel σ-field on K. For $f \in C(K)$ we have

$$\int_K |f| \, \mathrm{d}|\nu| \leqslant \|f\|_\infty \cdot |\nu|(K) \ ,$$

and hence f is ν-integrable. The function

$$\varphi\colon C(K) \to \mathbb{C} \quad \text{given by} \quad \varphi(f) = \int_K f \, \mathrm{d}\nu$$

is linear and bounded with $\|\varphi\| \leqslant \|\nu\|_1$. Thus, $\varphi \in M(K)$. Moreover,

(i) if ν is a signed measure, then $\varphi \in M^{\mathbb{R}}(K)$, and

(ii) if ν is a positive measure, then $\varphi \in M^+(K)$.

Remarks. Here $|\nu|$ is the *total variation measure* of ν: It is the smallest positive Borel measure μ that dominates ν in the sense that $|\nu(A)| \leqslant \mu(A)$ for all $A \in \mathcal{B}$. The *total variation* of ν is defined by $\|\nu\|_1 = |\nu|(K)$. A *signed measure* is a countably additive set-function $\nu\colon \mathcal{B} \to \mathbb{R}$. The expression *positive measure* simply means measure but is used for emphasis to distinguish it from complex and signed measures.

Theorem 13. (Riesz Representation Theorem) For every $\varphi \in M(K)$ there is a unique regular complex Borel measure ν on K that represents φ:

$$\varphi(f) = \int_K f \, \mathrm{d}\nu \qquad \text{for all } f \in C(K) \ .$$

Moreover, we have $\|\varphi\| = \|\nu\|_1$. Furthermore,

(i) if $\varphi \in M^{\mathbb{R}}(K)$, then ν is a signed measure, and

(ii) if $\varphi \in M^+(K)$, then ν is a positive measure.

Remark. Here *regularity* of ν means that for any Borel set A and $\varepsilon > 0$, there exist sets $E \subset A \subset U$ with E closed, U open and $|\nu|(U \setminus E) < \varepsilon$.

Corollary 14. The space of regular complex Borel measures on K with the total variation norm is a complex Banach space isometrically isomorphic to $M(K) = C(K)^*$.

The space of regular signed Borel measures on K is a real Banach space in the total variation norm, and it is isometrically isomorphic to $M^{\mathbb{R}}(K) \cong C^{\mathbb{R}}(K)^*$.

2 Weak topologies

Let X be a set and $\mathcal{F}$ be a family of functions such that each $f \in \mathcal{F}$ is a function $f\colon X \to Y_f$ where Y_f is a topological space. The *weak topology $\sigma(X, \mathcal{F})$ of X* is the smallest topology on X with respect to which every $f \in \mathcal{F}$ is continuous.

Remarks. 1. The family $\mathcal{S} = \{f^{-1}(U) : f \in \mathcal{F},\ U \subset Y_f \text{ an open set}\}$ is a sub-base for $\sigma(X, \mathcal{F})$. This means that $\sigma(X, \mathcal{F})$ is the topology generated by $\mathcal{S}$, *i.e.*, the smallest topology containing $\mathcal{S}$. Equivalently, the family of finite intersections of members of $\mathcal{S}$ is a base for $\sigma(X, \mathcal{F})$, *i.e.*, $\sigma(X, \mathcal{F})$ consists of arbitrary unions of finite intersections of members of $\mathcal{S}$. To spell this out, a set $V \subset X$ is open in the weak topology $\sigma(X, \mathcal{F})$ if and only if for all $x \in V$ there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathcal{F}$ and open sets $U_j \subset Y_{f_j}$, $1 \leqslant j \leqslant n$, such that

$$x \in \bigcap_{j=1}^{n} f_j^{-1}(U_j) \subset V \ .$$

To put it in another way, $V \subset X$ is open in the weak topology if and only if for all $x \in V$ there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathcal{F}$ and open neighbourhoods U_j of $f_j(x)$ in Y_{f_j}, $1 \leqslant j \leqslant n$, such that $\bigcap_{j=1}^{n} f_j^{-1}(U_j) \subset V$.
2. Suppose that $\mathcal{S}_f$ is a sub-base for the topology of Y_f for each $f \in \mathcal{F}$. Then $\{f^{-1}(U) : f \in \mathcal{F},\ U \in \mathcal{S}_f\}$ is a sub-base for $\sigma(X, \mathcal{F})$.
3. If Y_f is a Hausdorff space for each $f \in \mathcal{F}$ and $\mathcal{F}$ separates the points of X (for all $x \neq y$ in X there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$), then $\sigma(X, \mathcal{F})$ is Hausdorff.
4. Let $Y \subset X$ and set $\mathcal{F}_Y = \{f\restriction_Y \colon f \in \mathcal{F}\}$. Then $\sigma(X, \mathcal{F})\restriction_Y = \sigma(Y, \mathcal{F}_Y)$, *i.e.*, the subspace topology on Y induced by the weak topology $\sigma(X, \mathcal{F})$ of X is the same as the weak topology on Y defined by $\mathcal{F}_Y$.
5. (Universal property) Let Z be a topological space and $g \colon Z \to X$ a function. Then g is continuous if and only if $f \circ g \colon Z \to Y_f$ is continuous for every $f \in \mathcal{F}$. This universal property characterizes the weak topology.

Examples. 1. Let X be a topological space, $Y \subset X$ and $\iota \colon Y \to X$ be the inclusion map. Then the weak topology $\sigma(Y, \{\iota\})$ is the subspace topology of Y induced by X.
2. Let Γ be an arbitrary set, and let X_γ be a topological space for each $\gamma \in \Gamma$. Let X be the Cartesian product $\prod_{\gamma\in\Gamma} X_\gamma$. Thus

$$X = \{x : x \text{ is a function with domain } \Gamma, \text{ and } x(\gamma) \in X_\gamma \text{ for all } \gamma \in \Gamma\} \ .$$

For $x \in X$ we often write x_γ instead of $x(\gamma)$, and think of x as the "Γ-tuple" $(x_\gamma)_{\gamma\in\Gamma}$. For each γ we consider the function $\pi_\gamma \colon X \to X_\gamma$ given by $x \mapsto x(\gamma)$ (or $(x_\delta)_{\delta\in\Gamma} \mapsto x_\gamma$) called *evaluation at* γ or *projection onto* X_γ. The *product topology on* X is the weak topology $\sigma(X, \{\pi_\gamma : \gamma \in \Gamma\})$. Note that $V \subset X$ is open if and only if for all $x = (x_\gamma)_{\gamma\in\Gamma} \in V$ there exist $n \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_n \in \Gamma$ and open neighbourhoods U_j of x_{γ_j} in X_{γ_j} such that

$$\big\{y = (y_\gamma)_{\gamma\in\Gamma} : y_{\gamma_j} \in U_j \text{ for } 1 \leqslant j \leqslant n\big\} \subset V \ .$$

Proposition 1. Assume that X is a set, and for each $n \in \mathbb{N}$ we are given a metric space (Y_n, d_n) and a function $f_n \colon X \to Y_n$. Further assume that $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ separates the points of X. Then the weak topology $\sigma(X, \mathcal{F})$ of X is metrizable.

Remark. Without the assumption that $\mathcal{F}$ separates the points of X, we can conclude that $\sigma(X, \mathcal{F})$ is pseudo-metrizable.

Theorem 2. (Tychonov) The product of compact topological spaces is compact in the product topology.

Weak topologies on vector spaces. Let E be a real or complex vector space. Let F be a subspace of the space of all linear functionals on E that separates the points of E (for all $x \in E$, $x \neq 0$, there exists $f \in F$ such that $f(x) \neq 0$). Consider the weak topology $\sigma(E, F)$ on E. Note that $U \subset E$ is open if and only if for all $x \in U$ there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in F$ and $\varepsilon > 0$ such that

$$\{y \in E : |f_i(y - x)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \subset U \ .$$

For $f \in F$ define $p_f \colon E \to \mathbb{R}$ by $p_f(x) = |f(x)|$. Set $\mathcal{P} = \{p_f : f \in F\}$. Then $\mathcal{P}$ is a family of seminorms on E that separates the points of E, and the topology of the locally convex space $(E, \mathcal{P})$ is precisely $\sigma(E, F)$. In particular, $\sigma(E, F)$ is a Hausdorff topology with respect to which addition and scalar multiplication are continuous.

Lemma 3. Let E be as above. Let $f, g_1, g_2, \ldots, g_n$ be linear functionals on E. If $\bigcap_{j=1}^{n} \ker g_j \subset \ker f$, then $f \in \operatorname{span}\{g_1, \ldots, g_n\}$.

Proposition 4. Let E and F be as above. Then a linear functional f on E is $\sigma(E, F)$-continuous if and only if $f \in F$. In other words $(E, \sigma(E, F))^* = F$.

Note. Recall that $(E, \sigma(E, F))^*$ denotes the dual space of the locally convex space $(E, \sigma(E, F))$, *i.e.*, the space of linear functionals that are continuous with respect to $\sigma(E, F)$.

Examples. The following two examples of weak topologies on vector spaces are the central objects of interest in this chapter.

1. Let X be a normed space. The *weak topology on* X is the weak topology $\sigma(X, X^*)$ on X. Note that X^* separates the points of X by Hahn–Banach. We shall sometimes denote X with the weak topology by (X, w). Open sets in the weak topology are called *weakly open* or *w-open*. Note that $U \subset X$ is w-open if and only if for all $x \in U$ there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in X^*$ and $\varepsilon > 0$ such that

$$\{y \in X : |f_i(y - x)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \subset U \ .$$

2. Let X be a normed space. The *weak-star topology (or w^*-topology) on* X^* is the weak topology $\sigma(X^*, X)$ on X^*. Here we identify X with its image in X^{**} under the canonical embedding. We shall sometimes denote X^* with the w^*-topology by (X^*, w^*). Open sets in the weak-star topology are called *w^*-open*. Note that $U \subset X^*$ is w^*-open if and only if for all $f \in U$ there exist $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that

$$\{g \in X^* : |g(x_i) - f(x_i)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \subset U \ .$$

Properties. 1. (X, w) and (X^*, w^*) are locally convex spaces, so in particular they are Hausdorff, and addition and scalar multiplication are continuous.
2. $\sigma(X, X^*) \subset \|\cdot\|$-topology with equality if and only if $\dim X < \infty$.
3. $\sigma(X^*, X) \subset \sigma(X^*, X^{**}) \subset \|\cdot\|$-topology with equality in the second inclusion if and only if $\dim X < \infty$, and with equality in the first inclusion if and only if X is reflexive (*cf.* Proposition 5 below).
4. If Y is a subspace of X, then $\sigma(X, X^*)\restriction_Y = \sigma(Y, Y^*)$. This follows from Remark 4 on page 9 and from the Hahn–Banach theorem.

Similarly, we have $\sigma(X^{**}, X^*)|_X = \sigma(X, X^*)$, *i.e.*, the subspace topology on X induced by the w^*-topology of X^{**} is the weak topology of X. Thus, the canonical embedding $X \to X^{**}$ is a w-to-w^*-homeomorphism from X onto its image (as well as being an isometric isomorphism).

Proposition 5. Let X be a normed space. Then

(i) a linear functional f on X is weakly continuous (*i.e.*, continuous with respect to the weak topology on X) if and only if $f \in X^*$. Briefly, $(X, w)^* = X^*$;

(ii) a linear functional φ on X^* is w^*-continuous (*i.e.*, continuous with respect to the weak-star topology on X^*) if and only if $\varphi \in X$, *i.e.*, there exists $x \in X$ with $\varphi = \hat{x}$. Briefly, $(X^*, w^*)^* = X$.

It follows that on X^* the weak and weak-star topologies coincide, *i.e.*, we have $\sigma(X^*, X^{**}) = \sigma(X^*, X)$ if and only if X is reflexive.

Definition. Let X be a normed space.
A set $A \subset X$ is *weakly bounded* if $\{f(x) : x \in A\}$ is bounded for all $f \in X^*$. Equivalently, for each w-open ngbd U of 0, there exists $\lambda > 0$ such that $A \subset \lambda U$.
A set $B \subset X^*$ is *weak-star bounded* if $\{f(x) : f \in B\}$ is bounded for all $x \in X$. Equivalently, for each w^*-open ngbd U of 0, there exists $\lambda > 0$ such that $B \subset \lambda U$.

Principal of Uniform Boundedness (PUB). If X is a Banach space, Y is a normed space, $\mathcal{T} \subset \mathcal{B}(X, Y)$ is pointwise bounded ($\sup\{\|Tx\| : T \in \mathcal{T}\} < \infty$ for all $x \in X$), then $\mathcal{T}$ is uniformly bounded ($\sup\{\|T\| : T \in \mathcal{T}\} < \infty$).

Proposition 6. Let X be a normed space.

(i) Any weakly bounded set in X is bounded in norm.

(ii) If X is complete then any weak-star bounded set in X^* is bounded in norm.

Notation. 1. We write $x_n \xrightarrow{w} x$ and say x_n *weakly converges to* x if $x_n \to x$ in the weak topology (in some normed space X). This happens if and only if $\langle x_n, f\rangle \to \langle x, f\rangle$ for all $f \in X^*$. (See Remark 3 on page 5.)
2. We write $f_n \xrightarrow{w^*} f$ and say f_n w^**-converges to* f if $f_n \to f$ in the weak-star topology (in some dual space X^*). This happens if and only if $\langle x, f_n\rangle \to \langle x, f\rangle$ for all $x \in X$.

A consequence of PUB. Let X be a Banach space, Y a normed space, and (T_n) a sequence in $\mathcal{B}(X, Y)$ that converges pointwise to a function T. Then $T \in \mathcal{B}(X, Y)$, $\sup\|T_n\| < \infty$ and $\|T\| \leqslant \liminf\|T_n\|$.

Proposition 7. Let X be a normed space.

(i) If $x_n \xrightarrow{w} x$ in X, then $\sup\|x_n\| < \infty$ and $\|x\| \leqslant \liminf\|x_n\|$.

(ii) If $f_n \xrightarrow{w^*} f$ in X^* and X is complete, then $\sup\|f_n\| < \infty$ and $\|f\| \leqslant \liminf\|f_n\|$.

The Hahn–Banach Separation Theorems. Let $(X, \mathcal{P})$ be a locally convex space. Let C be a convex subset of X with $0 \in \operatorname{int} C$. Define

$$\mu_C \colon X \to \mathbb{R} \ , \qquad \mu_C(x) = \inf\{t > 0 : \ x \in tC\} \ .$$

This function is well-defined and is called the *Minkowski functional* or *gauge functional* of C.

Example. If X is a normed space and $C = B_X$, then $\mu_C = \|\cdot\|$.

Lemma 8. The function μ_C is a positive homogeneous, subadditive functional. Moreover, we have

$$\{x \in X : \ \mu_C(x) < 1\} \subset C \subset \{x \in X : \ \mu_C(x) \leqslant 1\}$$

with equality in the first inclusion when C is open and equality in the second inclusion when C is closed.

Remark. If C is symmetric in the case of real scalars, or balanced in the case of complex scalars, then μ_C is a seminorm. If in addition C is bounded, then μ_C is a norm.

Theorem 9. (Hahn–Banach separation theorem) Let $(X, \mathcal{P})$ be a locally convex space and C be an open convex subset of X with $0 \in C$. Let $x_0 \in X \setminus C$. Then there exists $f \in X^*$ such that $f(x_0) > f(x)$ for all $x \in C$ if the scalar field is $\mathbb{R}$, and $\operatorname{Re} f(x_0) > \operatorname{Re} f(x)$ for all $x \in C$ if the scalar field is $\mathbb{C}$.

Remark. From now on, for the rest of this chapter, we usually assume that the scalar field is $\mathbb{R}$. It is then straightforward to modify the statements of theorems, their proofs, etc, in the case of complex scalars.

Theorem 10. (Hahn–Banach separation theorem) Let $(X, \mathcal{P})$ be a locally convex space and A, B be non-empty disjoint convex subsets of X.

(i) If A is open, then there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(x) < \alpha \leqslant f(y)$ for all $x \in A$, $y \in B$.

(ii) If A is compact and B is closed, then there exists $f \in X^*$ such that $\sup_A f < \inf_B f$.

Theorem 11. (Mazur) Let C be a convex subset of a normed space X. Then $\overline{C}^{\|\cdot\|} = \overline{C}^{w}$, *i.e.*, the norm-closure and weak-closure of C are the same. It follows that C is norm-closed if and only if C is weakly closed.

Corollary 12. Assume that $x_n \xrightarrow{w} 0$ in a normed space X. Then for all $\varepsilon > 0$ there exists $x \in \operatorname{conv}\{x_n : \ n \in \mathbb{N}\}$ with $\|x\| < \varepsilon$.

Remark. It follows that there exist $p_1 < q_1 < p_2 < q_2 < \dots$ and convex combinations $\sum_{i=p_n}^{q_n} t_i x_i$ that converge to zero in norm. In some but not all cases a stronger conclusion can be obtained.

Theorem 13. (Banach–Alaoglu) The dual ball B_{X^*} is w^*-compact for any normed space X.

Proposition 14. Let X be a normed space and K be a compact Hausdorff space.

(i) X is separable if and only if (B_{X^*}, w^*) is metrizable.

(ii) $C(K)$ is separable if and only if K is metrizable.

Remarks. 1. If X is separable, then (B_{X^*}, w^*) is a compact metric space. In particular, B_{X^*} is w^*-sequentially compact.
2. If X is separable, then X^* is w^*-separable. It is an easy consequence of Mazur's theorem that X is separable if and only if X is weakly separable. Thus, the previous statement reads: if X is w-separable, then X^* is w^*-separable. The converse is false, *e.g.,* for $X = \ell_\infty$ (see the remark following Goldstine's theorem).
3. The proof shows that any compact Hausdorff space K is a subspace of $(B_{C(K)^*}, w^*)$.
4. The proof shows that every normed space X embeds isometrically into $C(K)$ for some compact Hausdorff space. In particular, this holds with $K = (B_{X^*}, w^*)$ (*cf.* Theorem 19 below).

Proposition 15. X^* is separable if and only if (B_X, w) is metrizable.

Theorem 16. (Goldstine) $\overline{B_X}^{w^*} = B_{X^{**}}$, *i.e.*, the w^*-closure of the unit ball B_X of a normed space X in the second dual X^{**} is $B_{X^{**}}$.

Remark. It follows from Goldstine that $\overline{X}^{w^*} = X^{**}$. Thus, if X is separable, then X^{**} is w^*-separable. For example, ℓ_∞^* is w^*-separable.

Theorem 17. Let X be a Banach space. Then TFAE.

(i) X is reflexive.

(ii) (B_X, w) is compact.

(iii) X^* is reflexive.

Remark. It follows that if X is separable and reflexive, then (B_X, w) is a compact metric space, and hence sequentially compact.

Lemma 18. For every non-empty compact metric space K there is a continuous surjection $\varphi\colon \Delta \to K$, where $\Delta = \{0,1\}^{\mathbb{N}}$ with the product topology.

Note. Δ is homeomorphic via the map $(\varepsilon_i) \mapsto \sum_i (2\varepsilon_i)3^{-i}$ to the middle-third Cantor set.

Theorem 19. Every separable Banach space embeds isometrically into $C[0,1]$. Thus the separable space $C[0,1]$ is isometrically universal for the class of separable Banach spaces.

3 Schauder bases

Definition. A *Schauder basis* or simply *basis* of a Banach space X is a sequence (e_n) in X such that for all $x \in X$ there exists a unique sequence (a_n) of scalars such that x is the norm-convergent sum $x = \sum_{n=1}^{\infty} a_n e_n$.

Notation. For a sequence (e_n) in X, we let $[e_n] = \overline{\text{span}}\{e_n : n \in \mathbb{N}\}$.

Note. If (e_n) is a basis of X, then (e_n) is linearly independent and $X = [e_n]$. Thus, X is separable and infinite-dimensional, and it can identified with the sequence space $\big\{(a_n) \in \mathbb{R}^{\mathbb{N}} \text{ (or } \mathbb{C}^{\mathbb{N}}) : \sum_{n=1}^{\infty} a_n e_n \text{ is convergent in } X\big\}$.

Definition. A sequence (e_n) in a Banach space X is called a *basic sequence* if it is a basis of its closed linear span $[e_n]$.

Theorem 1. Let (e_n) be a sequence in a Banach space X. Then (e_n) is a basis of X if and only if all of the following three conditions hold:

(i) $e_n \neq 0$ for all $n \in \mathbb{N}$.

(ii) There exists $C \geqslant 1$ such that

$$\Big\| \sum_{i=1}^{m} a_i e_i \Big\| \leqslant C \Big\| \sum_{i=1}^{n} a_i e_i \Big\|$$

for all scalar sequences (a_i) and all $1 \leqslant m \leqslant n$ in $\mathbb{N}$.

(iii) $X = [e_n]$.

It follows that (e_n) is a basic sequence if and only if conditions (i) and (ii) hold.

Remark. If (e_n) is a basis of X, then the projections $P_n \colon X \to X$ defined by $P_n\big(\sum_{i=1}^{\infty} a_i e_i\big) = \sum_{i=1}^{n} a_i e_i$ are called the *basis projections*. By the above theorem, $\{P_n : n \in \mathbb{N}\}$ is uniformly bounded. The constant $C = \sup_{n\in\mathbb{N}} \|P_n\|$ is called the *basis constant* of (e_n), and it is the least C that satisfies condition (ii) above. A basis with basis constant 1 is called a *monotone basis*.

Examples. 1. The *Schauder system* is a monotone basis of $C[0,1]$.

2. The *unit vector basis* is a monotone basis of ℓ_p, $1 \leqslant p < \infty$, and of c_0.

3. The *Haar system* is a monotone basis of $L_p[0,1]$ for $1 \leqslant p < \infty$.

Remark. A famous result of Per Enflo shows that there are separable infinite-dimensional Banach spaces without bases. However, we have the following.

Theorem 2. (Mazur) Every infinite-dimensional Banach space contains a basic sequence. More precisely, for any $\varepsilon > 0$, every infinite-dimensional Banach space contains a normalised basic sequence with basis constant at most $1 + \varepsilon$.

Lemma 3. Let F be a finite-dimensional subspace of an infinite-dimensional Banach space X, and let $\varepsilon > 0$. Then there exists $y \in S_X$ such that $\|x\| \leqslant (1+\varepsilon)\|x + \lambda y\|$ for all $x \in F$ and all scalars λ.

Remark. The proof of the lemma shows that there is a finite-codimensional subspace Y of X such that any $y \in S_Y$ works.

Dual bases. Let (e_n) be a basis of a Banach space X, let P_n, $n \in \mathbb{N}$, be the corresponding basis projections, and let C be the basis constant. For $n \in \mathbb{N}$, define the n^{th} *coordinate functional* e_n^* by

$$\big\langle \sum a_i e_i, e_n^* \big\rangle = a_n \ .$$

Then $e_n^* \in X^*$ with $\frac{1}{\|e_n\|} \leqslant \|e_n^*\| \leqslant \frac{2C}{\|e_n\|}$. Moreover,

$$P_n^*(x^*) = \sum_{i=1}^{n} x^*(e_i) e_i^* \qquad (x^* \in X^*,\ n \in \mathbb{N}).$$

In particular, $P_n^*(e_i^*) = 0$ if $n < i$ and $P_n^*(e_i^*) = e_i^*$ if $i \leqslant n$, and thus $P_m^*(\sum_{i=1}^n a_i e_i^*) = \sum_{i=1}^m a_i e_i^*$ for $m \leqslant n$ and scalars (a_i). It follows that (e_n^*) is a basic sequence in X^* with basis constant at most C, whose n^{th} basis projection is the restriction of P_n^* to $[e_k^*]$. We call (e_i^*) the *dual basis* or *biorthogonal sequence* of (e_i).

Examples. 1. If (e_n) is the unit vector basis of ℓ_p $(1 \leqslant p < \infty)$ or of c_0, then (e_n^*) is the unit vector basis of $\ell_p^* = \ell_q$ (where q is the conjugate index of p) or of ℓ_1, respectively. Note that (e_n^*) is not a basis of $\ell_1^* = \ell_\infty$.

2. The dual basis of the Haar basis $(h_n)_{n \geqslant 0}$ of $L_p[0,1]$ $(1 \leqslant p < \infty)$ is the (suitably scaled) Haar basis $(\lambda_n h_n)_{n \geqslant 0}$ of $L_p[0,1]^* = L_q[0,1]$ (where q is the conjugate index of p).

Remark. Although in general, the dual basis (e_n^*) of a basis (e_n) of a Banach space X need not be a basis of X^*, every $x^* \in X^*$ does have a unique representation $\sum_{n=1}^{\infty} x^*(e_n) e_n^*$ with the sum converging to x^* in the w^*-topology.

Equivalence of bases and perturbation. We begin with a simple result.

Proposition 4. For two basic sequences (e_n) and (f_n) in possibly different Banach spaces the following conditions are equivalent.

(i) $\sum a_i e_i$ converges if and only if $\sum a_i f_i$ converges for all (a_i).

(ii) The map $T(e_n) = f_n$ extends to an isomorphism $T\colon [e_n] \to [f_n]$.

(iii) There exist constants $A > 0$ and $B > 0$ such that

$$A\Big\| \sum a_i e_i \Big\| \leqslant \Big\| \sum a_i f_i \Big\| \leqslant B \Big\| \sum a_i e_i \Big\|$$

for all $(a_i) \in \mathrm{c}_{00}$.

Definition. Two basic sequences (e_n) and (f_n) in possibly different Banach spaces are *equivalent*, written $(e_n) \sim (f_n)$, if any of the conditions (i), (ii) or (iii) in the proposition above is satisfied. More precisely, we say that (e_n) and (f_n) are C-*equivalent*, written $(e_n) \underset{C}{\sim} (f_n)$, if (iii) holds with $B/A \leqslant C$, or equivalently, if (ii) holds with $\|T\|\|T^{-1}\| \leqslant C$.