More measure theory and the dual of $C(K)$

Introduction. This is an expanded version of what was covered in lectures on the dual of $C(K)$ spaces. The aim is to provide all the necessary background material mainly from measure theory and to some extent from topology. The additional material here does not form part of the examinable material.

Complex measures. Let Ω be a set and $\mathcal{F}$ be a σ-field on Ω. A *complex measure on* $\mathcal{F}$ is a countably additive set function $\nu\colon \mathcal{F} \to \mathbb{C}$. The *total variation measure* $|\nu|$ of ν is defined at $A \in \mathcal{F}$ as follows.

$$|\nu|(A) = \sup\Big\{ \sum_{k=1}^{n} |\nu(A_k)| : A = \bigcup_{k=1}^{n} A_k \text{ is a measurable partition of } A \Big\} .$$

(*Measurable partition* means that $A_k \in \mathcal{F}$ for all k, and $A_j \cap A_k = \emptyset$ for all $j \neq k$.) It is easy to check that $|\nu|\colon \mathcal{F} \to [0, \infty]$ is a positive measure. (The expression *positive measure* simply means measure but is used for emphasis to distinguish it from complex and signed measures.) It is also straightforward to verify that $|\nu|$ is the smallest positive measure that dominates ν. We will soon see that $|\nu|$ is in fact a finite measure, *i.e.*, that $|\nu|(\Omega) < \infty$ (see the remarks following Theorem 1.) We define the *total variation* $\|\nu\|_1$ *of* ν to be $|\nu|(\Omega)$.

Continuity of measure. A complex measure ν on $\mathcal{F}$ is continuous in the following sense. Given a sequence (A_n) in $\mathcal{F}$,

(i) if $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$, then $\nu\big(\bigcup A_n\big) = \lim \nu(A_n)$, and

(ii) if $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$, then $\nu\big(\bigcap A_n\big) = \lim \nu(A_n)$.

Signed measures. Let $(\Omega, \mathcal{F})$ be as before. A *signed measure on* $\mathcal{F}$ is a countably additive set function $\nu\colon \mathcal{F} \to \mathbb{R}$. Every signed measure is in particular a complex measure.

Theorem 1. Let $\nu\colon \mathcal{F} \to \mathbb{R}$ be a signed measure on the σ-field $\mathcal{F}$. Then there is a measurable partition $\Omega = P \cup N$ of Ω such that $\nu(A) \geqslant 0$ for all $A \subset P$, $A \in \mathcal{F}$ and $\nu(A) \leqslant 0$ for all $A \subset N$, $A \in \mathcal{F}$.

Remarks. 1. The partition $\Omega = P \cup N$ is called the *Hahn decomposition* of Ω (or of ν).
2. Define $\nu^+(A) = \nu(A \cap P)$ and $\nu^-(A) = -\nu(A \cap N)$ for $A \in \mathcal{F}$. Then ν^+ and ν^- are finite positive measures satisfying $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$. These properties uniquely define ν^+ and ν^-. This unique way of writing the signed measure ν as the difference of two finite positive measures is called the *Jordan decomposition of* ν.
3. Let $\nu\colon \mathcal{F} \to \mathbb{C}$ be a complex measure. Then $\mathrm{Re}(\nu)$ and $\mathrm{Im}(\nu)$ are signed measures, so they have Jordan decompositions $\nu_1 - \nu_2$ and $\nu_3 - \nu_4$, respectively. We then obtain the expression $\nu = \nu_1 - \nu_2 + \mathrm{i}\nu_3 - \mathrm{i}\nu_4$ called the Jordan decomposition of ν. It follows that $\nu_k \leqslant |\nu|$ for all k, and $|\nu| \leqslant \nu_1 + \nu_2 + \nu_3 + \nu_4$. From this we deduce that $|\nu|$ is a finite measure.

4. Let ν be a signed measure with Jordan decomposition $\nu^+ - \nu^-$ (as in **2.** above). Then for each $A \in \mathcal{F}$ we have $\nu^+(A) = \sup\{\nu(B) : B \subset A,\ B \in \mathcal{F}\}$. This motivates the proof of Theorem 1.

Absolute continuity. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\nu\colon \mathcal{F} \to \mathbb{C}$ be a complex measure. We say ν *is absolutely continuous with respect to* μ, and write $\nu \ll \mu$, if $\nu(A) = 0$ whenever $A \in \mathcal{F}$ and $\mu(A) = 0$.

Remarks. 1. If $\nu \ll \mu$, then $|\nu| \ll \mu$. It follows that if $\nu = \nu_1 - \nu_2 + \mathrm{i}\nu_3 - \mathrm{i}\nu_4$ is the Jordan decomposition of ν, then $\nu_k \ll \mu$ for all k.
2. If $\nu \ll \mu$, then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for every $A \in \mathcal{F}$ we have $\mu(A) < \delta \implies |\nu(A)| < \varepsilon$.

Example. Let $f \in L_1(\mu)$ and define $\nu(A) = \int_A f \,\mathrm{d}\mu$ for $A \in \mathcal{F}$. Then by dominated convergence, ν is a complex or signed measure, and $\nu \ll \mu$.

Definition. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A set $A \in \mathcal{F}$ is called *σ-finite* if there is a sequence (A_n) in $\mathcal{F}$ such that $A = \bigcup_n A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. We say that the measure μ (or the measure space $(\Omega, \mathcal{F}, \mu)$) is σ-finite if Ω is a σ-finite set (equivalently every $A \in \mathcal{F}$ is σ-finite).

Theorem 2. (Radon–Nikodym) Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $\nu\colon \mathcal{F} \to \mathbb{C}$ be a complex measure on $\mathcal{F}$ which is absolutely continuous with respect to μ. Then there exists a unique $f \in L_1(\mu)$ such that $\nu(A) = \int_A f \,\mathrm{d}\mu$ for all $A \in \mathcal{F}$. Moreover, f takes values in $\mathbb{C}/\mathbb{R}/\mathbb{R}^+$ according to whether ν is a complex/signed/positive measure, respectively.

Remarks. 1. The proof shows that for an arbitrary complex measure ν on $\mathcal{F}$ there is a decomposition $\nu = \nu_1 + \nu_2$, where ν_1 is absolutely continuous with respect to μ (and hence of the form $\int_A f \,\mathrm{d}\mu$ for some $f \in L_1(\mu)$), and ν_2 is *orthogonal* to μ, written $\nu_2 \perp \mu$, which means that there is a measurable partition $\Omega = P \cup N$ of Ω with $\mu(P) = 0$ and $\nu_2(A) = 0$ for all $A \subset N$. This decomposition is called the *Lebesgue decomposition of ν with respect to μ.*
2. The unique $f \in L_1(\mu)$ in the theorem is called the *Radon–Nikodym derivative of ν with respect to μ* and is denoted by $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. It can be shown that a measurable function g is ν-integrable if and only if $g \cdot \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is μ-integrable in which case we have

$$\int_\Omega g \,\mathrm{d}\nu = \int_\Omega g \cdot \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,\mathrm{d}\mu \ .$$

Integration with respect to complex measures. Let Ω be a set, $\mathcal{F}$ be a σ-field on Ω, and ν be a complex measure on $\mathcal{F}$. A measurable function $f\colon \Omega \to \mathbb{C}$ is *ν-integrable* if $\int_\Omega |f| \,\mathrm{d}|\nu| < \infty$. In that case we define

$$\int_\Omega f \,\mathrm{d}\nu = \int_\Omega f \,\mathrm{d}\nu_1 - \int_\Omega f \,\mathrm{d}\nu_2 + \mathrm{i}\int_\Omega f \,\mathrm{d}\nu_3 - \mathrm{i}\int_\Omega f \,\mathrm{d}\nu_4 \ ,$$

where $\nu = \nu_1 - \nu_2 + \mathrm{i}\nu_3 - \mathrm{i}\nu_4$ is the Jordan decomposition of ν. Note that f is ν-integrable if and only if f is ν_k-integrable for each k. The following properties are easy to check:
1. $\int_\Omega \mathbf{1}_A \,\mathrm{d}\nu = \nu(A)$ for all $A \in \mathcal{F}$.
2. Linearity: given ν-integrable functions f, g and complex numbers a, b, the function $af + bg$ is ν-integrable, and $\int_\Omega (af + bg) \,\mathrm{d}\nu = a\int_\Omega f \,\mathrm{d}\nu + b\int_\Omega g \,\mathrm{d}\nu$.

3. Dominated convergence (D.C.): let f_n, $n \in \mathbb{N}$, be ν-integrable functions that converge a.e. to a measurable function f. Assume that there exists $g \in L_1(|\nu|)$ with $|f_n| \leqslant g$ for all n. Then f is ν-integrable and $\int_\Omega f_n \,\mathrm{d}\nu \to \int_\Omega f \,\mathrm{d}\nu$.

4. $\left| \int_\Omega f \,\mathrm{d}\nu \right| \leqslant \int_\Omega |f| \,\mathrm{d}|\nu|$ for all ν-integrable f.

***C*(*K*) spaces.** For the rest of these notes, K will be a compact Hausdorff space. We shall be interested in the following spaces and sets.

$C(K) = \{f \colon K \to \mathbb{C} : \ f \text{ is continuous}\}$

$C^{\mathbb{R}}(K) = \{f \colon K \to \mathbb{R} : \ f \text{ is continuous}\}$

$C^+(K) = \{f \in C(K) : \ f \geqslant 0 \text{ on } K\}$

$M(K) = C(K)^* = \{\varphi \colon C(K) \to \mathbb{C} : \ \varphi \text{ is linear and continuous}\}$

$M^{\mathbb{R}}(K) = \{\varphi \in M(K) : \ \varphi(f) \in \mathbb{R} \text{ for all } f \in C^{\mathbb{R}}(K)\}$

$M^+(K) = \{\varphi \colon C(K) \to \mathbb{C} : \ \varphi \text{ is linear and } \varphi(f) \geqslant 0 \text{ for all } f \in C^+(K)\}$

Some comments on the above list:

(i) $C(K)$ is a complex Banach space in the uniform norm: $\|f\|_\infty = \sup_K |f|$.

(ii) $C^{\mathbb{R}}(K)$ is a real Banach space in the uniform norm.

(iii) $C^+(K)$ is a subset of $C(K)$. By "$f \geqslant 0$ on K" we of course mean that $f(x) \geqslant 0$ for all $x \in K$.

(iv) $M(K)$ is the dual space of $C(K)$, and so it is a complex Banach space in the operator norm.

(v) $M^{\mathbb{R}}(K)$ is a closed, real-linear subspace of $M(K)$. We shall soon see that it can be identified with the dual space of $C^{\mathbb{R}}(K)$.

(vi) Elements of $M^+(K)$ are called *positive linear functionals*. We shall see that these are automatically continuous.

Note. Our aim is to describe the dual spaces $M(K)$ and $M^{\mathbb{R}}(K)$. The main conclusion of the next lemma is that it is sufficient to focus on $M^+(K)$.

Lemma 3.

(i) $\forall\, \varphi \in M(K)$ $\exists$ unique $\varphi_1, \varphi_2 \in M^{\mathbb{R}}(K)$ such that $\varphi = \varphi_1 + \mathrm{i}\varphi_2$.

(ii) $\varphi \mapsto \varphi{\restriction}_{C^{\mathbb{R}}(K)} \colon M^{\mathbb{R}}(K) \to \big(C^{\mathbb{R}}(K)\big)^*$ is an isometric isomorphism.

(iii) $M^+(K) \subset M^{\mathbb{R}}(K)$ and $M^+(K) = \big\{\varphi \in M(K) : \ \|\varphi\| = \varphi(\mathbf{1}_K)\big\}$.

(iv) For every $\varphi \in M^{\mathbb{R}}(K)$ there exist unique $\varphi^+, \varphi^- \in M^+(K)$ such that

$$\varphi = \varphi^+ - \varphi^- \qquad \text{and} \qquad \|\varphi\| = \|\varphi^+\| + \|\varphi^-\| \ .$$

Topological preliminaries. 1. A compact Hausdorff space is *normal*: given disjoint closed subsets E and F of K, there exist disjoint open subsets U and V of K such that $E \subset U$ and $F \subset V$. Equivalently, given $E \subset U \subset K$ with E closed and U open, there is an open set $V \subset K$ such that $E \subset V \subset \overline{V} \subset U$.

2. Urysohn's lemma: given disjoint closed subsets E and F of K, there is a continuous function $f\colon K \to [0,1]$ such that $f = 0$ on E and $f = 1$ on F.

3. The notation $f \prec U$ will mean that U is an open subset of K, f is a continuous function $K \to [0,1]$, and the *support of* f,

$$\operatorname{supp} f = \overline{\{x \in K :\, f(x) \neq 0\}}$$

is contained in U. The notation $E \prec f$ will mean that E is a closed subset of K, f is a continuous function $K \to [0,1]$, and $f = 1$ on E.

Note that Urysohn's lemma is equivalent to the following statement. Given $E \subset U \subset K$ with E closed and U open, there is an f with $E \prec f \prec U$.

Lemma 4. Let E and $U_1, \ldots, U_n$ be subsets of K with E closed, U_j open, and $E \subset \bigcup_{j=1}^n U_j$. Then

(i) there exist open sets $V_1, \ldots, V_n$ with $E \subset \bigcup_{j=1}^n V_j$ and $\overline{V}_j \subset U_j$ for all j;

(ii) there exist functions $f_j \prec U_j$ such that $0 \leqslant \sum_{j=1}^n f_j \leqslant 1$ on K, and $\sum_{j=1}^n f_j = 1$ on E.

Borel measures and regularity. Let X be a Hausdorff topological space and $\mathcal{G}$ be the collection of open subsets of X. The *Borel σ-field on X* is defined to be $\mathcal{B} = \sigma(\mathcal{G})$, the σ-field generated by $\mathcal{G}$, *i.e.*, the smallest σ-field on X containing $\mathcal{G}$. Equivalently, $\mathcal{B}$ is the intersection of all σ-fields on X that contain $\mathcal{G}$. Members of $\mathcal{B}$ are called *Borel sets*.

A *Borel measure on X* is a measure on $\mathcal{B}$. Given a (positive) Borel measure μ on X, we say μ is *regular* if the following hold:

(i) $\mu(E) < \infty$ for all compact $E \subset X$;

(ii) $\mu(A) = \inf\{\mu(U) :\, A \subset U \in \mathcal{G}\}$ for all $A \in \mathcal{B}$;

(iii) $\mu(U) = \sup\{\mu(E) :\, E \subset U, E \text{ compact}\}$ for all $U \in \mathcal{G}$.

A complex Borel measure ν on X is defined to be *regular* if $|\nu|$ is a regular measure.

Note that if X is compact Hausdorff, then a Borel measure μ on X is regular if and only if

$$\mu(X) < \infty \quad \text{and} \quad \mu(A) = \inf\{\mu(U) :\, A \subset U \in \mathcal{G}\} \quad \forall\, A \in \mathcal{B}$$

which in turn is equivalent to

$$\mu(X) < \infty \quad \text{and} \quad \mu(A) = \sup\{\mu(E) :\, E \subset A,\ E \text{ closed}\} \quad \forall\, A \in \mathcal{B}\ .$$

Note. Let ν be a complex Borel measure on K. For $f \in C(K)$ we have

$$\int_K |f|\, \mathrm{d}|\nu| \leqslant \|f\|_\infty \cdot |\nu|(K)\ ,$$

and hence f is ν-integrable. The function

$$\varphi\colon C(K) \to \mathbb{C} \quad \text{given by} \quad \varphi(f) = \int_K f \, \mathrm{d}\nu$$

is linear and bounded with $\|\varphi\| \leqslant \|\nu\|_1$. Thus, $\varphi \in M(K)$. Moreover,

(i) if ν is a signed measure, then $\varphi \in M^{\mathbb{R}}(K)$, and

(ii) if ν is a positive measure, then $\varphi \in M^{+}(K)$.

Theorem 5. (Riesz Representation Theorem) For every $\varphi \in M^{+}(K)$ there is a unique regular Borel measure μ on K that represents φ:

$$\varphi(f) = \int_K f \, \mathrm{d}\mu \qquad \text{for all } f \in C(K) \ .$$

Moreover, we have $\mu(K) = \|\varphi\|$.

Corollary 6. (Riesz Representation Theorem) For every $\varphi \in M(K)$ there is a unique regular complex Borel measure ν on K that represents φ:

$$\varphi(f) = \int_K f \, \mathrm{d}\nu \qquad \text{for all } f \in C(K) \ .$$

Moreover, we have $\|\varphi\| = \|\nu\|_1$, and if $\varphi \in M^{\mathbb{R}}(K)$ then ν is a signed measure.

Corollary 7. The space of regular complex Borel measures on K is a complex Banach space in the total variation norm, and it is isometrically isomorphic to $M(K) = C(K)^*$. The space of regular signed Borel measures on K is a real Banach space in the total variation norm, and it is isometrically isomorphic to $M^{\mathbb{R}}(K) \cong \big(C^{\mathbb{R}}(K)\big)^*$.