

Part II Linear Analysis

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Prerequisites.

- Linear Algebra (just a little bit: vector spaces, subspaces, quotient spaces, linear maps, ...)
- Analysis II (except differentiation), Topological Spaces (except manifolds)

Books.

- B. Bollobás: Linear Analysis (CUP)
- G. J. O. Jameson: Topology and Normed Spaces (Chapman and Hall)
- G. R. Allan: Introduction to Banach Spaces and Algebras (OUP)
- W. Rudin: Real and Complex Analysis (McGraw-Hill)

1 Normed spaces and bounded linear maps

Norms. Let X be a real or complex vector space. A *norm* on X is a function $\|\cdot\|: X \rightarrow \mathbb{R}$ satisfying the following:

- (i) $\|x\| \geq 0$ for all $x \in X$ with equality iff $x = 0$ (*positivity*);
- (ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and scalars λ (*homogeneity*);
- (iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (*triangle-inequality*).

The quantity $\|x\|$ is called the *norm* or *length* of x .

Normed spaces. A *normed space* is a pair $(X, \|\cdot\|)$ where X is a real or complex vector space and $\|\cdot\|$ is a norm on X .

Examples of normed spaces.

1. $\ell_2^n = (\mathbb{R}^n, \|\cdot\|_2)$ or $(\mathbb{C}^n, \|\cdot\|_2)$ where

$$\|x\|_2 = \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \quad \text{for } x = (x_1, \dots, x_n)$$

called the ℓ_2 -norm or *euclidean norm*.

2. $\ell_1^n = (\mathbb{R}^n, \|\cdot\|_1)$ or $(\mathbb{C}^n, \|\cdot\|_1)$ where

$$\|x\|_1 = \sum_{j=1}^n |x_j| \quad \text{for } x = (x_1, \dots, x_n)$$

called the ℓ_1 -norm.

3. $\ell_\infty^n = (\mathbb{R}^n, \|\cdot\|_\infty)$ or $(\mathbb{C}^n, \|\cdot\|_\infty)$ where

$$\|x\|_\infty = \max_{1 \leq j \leq n} |x_j| \quad \text{for } x = (x_1, \dots, x_n)$$

called the ℓ_∞ -norm or *sup-norm*.

Induced metric. Given a normed space X , the norm induces a metric on X given by

$$d(x, y) = \|x - y\| \quad (x, y \in X).$$

In turn, this metric induces a topology on X called the *norm topology*. This allows us to talk about open and closed sets, convergence, continuity, etc.

Note. The algebraic operations are continuous:

- If $x_n \rightarrow x$ and $y_n \rightarrow y$ in X , then $x_n + y_n \rightarrow x + y$.
- If $x_n \rightarrow x$ and scalars $\lambda_n \rightarrow \lambda$, then $\lambda_n x_n \rightarrow \lambda x$.

The norm is also continuous: $|\|x\| - \|y\|| \leq \|x - y\|$ for all $x, y \in X$.

Banach spaces. A *Banach space* is a complete normed space, *i.e.*, a normed space that is complete in its induced metric. *E.g.*, ℓ_2^n , ℓ_1^n and ℓ_∞^n are complete.

Unit ball. In a normed space X , a useful object is the *unit ball*

$$B_X = \{x \in X : \|x\| \leq 1\}.$$

Remarks. 1. B_X determines the norm: $\|x\| = \inf\{t \geq 0 : x \in tB_X\}$.

2. B_X is a closed, bounded, convex and symmetric ($x \in B_X \Rightarrow -x \in B_X$) neighbourhood of 0.

3. If $B \subset \mathbb{R}^n$ is a closed, bounded, convex and symmetric neighbourhood of 0, then B is the unit ball of $\mathbb{R}^n$ with respect to a (unique) norm.

4. The ‘shape’ of B_X is often important (‘Geometry of Banach spaces’).

Examples of normed spaces (continued).

4. $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ or $(\mathbb{C}^n, \|\cdot\|_p)$ for $1 \leq p < \infty$ where

$$\|x\|_p = \left(\sum_{j=1}^n |x_j|^p \right)^{1/p} \quad \text{for } x = (x_1, \dots, x_n)$$

called the ℓ_p -norm. The triangle inequality in ℓ_p is called Minkowski’s inequality and it is not obvious (see below).

Now, let S be the vector space of all scalar sequences with pointwise operations: $(x_n) + (y_n) = (x_n + y_n)$ and $\lambda(x_n) = (\lambda x_n)$.

5. $\ell_1 = \{(x_n) \in S : \sum_n |x_n| < \infty\}$ with the ℓ_1 -norm $\|(x_n)\|_1 = \sum_n |x_n|$.