1. Let (x_n) be a normalised basic sequence in some Banach space. Show that (x_n) dominates the unit vector basis (e_n) of c_0: there exists $B \geqslant 1$ such that $\|\sum a_i e_i\|_\infty \leqslant B\|\sum a_i x_i\|$ for all $(a_i) \in c_{00}$.

2. Show that $C[0,1]$ has a Schauder basis consisting of polynomials.

3. Let (e_n) be the unit vector basis of ℓ_1. Show that for any permutation π of $\mathbb{N}$, the sequence $(e_{\pi(n)})$ is a basis of ℓ_1. Now, for each $n \in \mathbb{N}$, set $x_n = e_n - e_{n-1}$ ($e_0 = 0$). Prove that (x_n) is a basis of ℓ_1 but that $(x_{\pi(n)})$ is not a basis for certain permutations π of $\mathbb{N}$.

4. The *Schreier space* X is the completion of the space c_{00} of eventually zero scalar sequences with respect to the norm

$$\|(a_i)\| = \sup\left\{ \sum_{i=1}^{k} |a_{n_i}| : k \leqslant n_1 < n_2 < \cdots < n_k \right\} .$$

Show that the unit vector basis (e_n) is a normalised, monotone basis of X.

Let (u_n) be a normalised block basis of (e_n) such that $\|u_n\|_\infty \to 0$. Show that some subsequence of (u_n) is equivalent to the unit vector basis of c_0. Deduce that (e_n) is weakly null.

Show that X is c_0-*saturated*: every closed infinite-dimensional subspace of X contains a further subspace isomorphic to c_0.

5. An operator $T\colon X \to Y$ between Banach spaces is *strictly singular* if it is not an isomorphism on any infinite-dimensional subspace. Show that the sum of two strictly singular operators is strictly singular. [Hint: Construct a suitable basic sequence.] Deduce that the set $\mathcal{S}(X,Y)$ of all strictly singular operators $X \to Y$ is a closed subspace of $\mathcal{B}(X,Y)$ with the ideal property.

6. Let $1 \leqslant p < q < \infty$. Prove that $\mathcal{B}(\ell_p, \ell_q) = \mathcal{S}(\ell_p, \ell_q)$.

7. Let $s_n = \sum_{i=1}^{n} e_i$. Show that (s_n) is a basis of c_0 with basis constant 2. This is called the *summing basis* of c_0.

Let X be a non-reflexive Banach space. Show that there is a sequence (x_n) in S_X with a w^*-accumulation point $\varphi \in X^{**} \setminus X$. Deduce that some subsequence (y_n) of (x_n) is a basic sequence that dominates the summing basis.

8. Use the Pełczyński Decomposition Method to show that $\ell_\infty \sim L_\infty[0,1]$. [You may assume that $L_\infty[0,1]$ is complemented in every superspace.]

9. Let $T\colon X \to Y$ be a linear bijection between (real) vector spaces and let C be a convex subset of X. Show that $T(\operatorname{Ext} C) = \operatorname{Ext} T(C)$. Now assume that $T\colon X \to Y$ is a continuous linear map between locally convex spaces and C is a compact convex subset of X. Show that $T(\operatorname{Ext} C) \supset \operatorname{Ext} T(C)$.

10. Show that $B_{\ell_\infty} = \overline{\operatorname{conv}}\operatorname{Ext} B_{\ell_\infty}$ but that $B_{C[0,1]^*} \neq \overline{\operatorname{conv}}\operatorname{Ext} B_{C[0,1]^*}$.

11. Let A be a Banach algebra. Let $p \in A$ be such that $p^2 = p$ (such an element of called an *idempotent*). Show that if p is in the closure of an ideal J of A, then p does in fact belong to J.

12. Let Δ be the closed unit disc in $\mathbb{C}$. Show that one can choose a sequence (D_n) of non-overlapping open discs in the interior of Δ such that $\sum_n \operatorname{diam}(D_n) < \infty$ and $\bigcup_n D_n$ is dense in Δ. Set $K = \Delta \setminus \bigcup_n D_n$ (the *Swiss Cheese*). Show that the following formula defines a non-zero, bounded linear functional on $C(K)$.

$$\theta(f) = \int_{\partial\Delta} f(z)\,\mathrm{d}z - \sum_{n=1}^{\infty} \int_{\partial D_n} f(z)\,\mathrm{d}z$$

Deduce that $\mathcal{R}(K) \neq A(K)$.