1. Let K be a compact Hausdorff space, and (f_n) be a bounded sequence in $C(K)$. Show that (f_n) is weakly null if and only if it is pointwise null.

2. Let $T\colon X \to Y$ be a linear map between normed spaces. Show that the following are equivalent.

(i) T is norm-to-norm continuous.
(ii) T is weak-to-weak continuous.
(iii) T is norm-to-weak continuous.

Show further that T is weak-to-norm continuous if and only if T is bounded and of finite-rank.

3. Let $T\colon Y^* \to X^*$ be a bounded linear map between dual spaces. Show that the following are equivalent.

(i) T is w^*-to-w^* continuous.
(ii) There is a bounded linear map $S\colon X \to Y$ with $S^* = T$.
(iii) $T^*(X) \subset Y$.

4. When is the canonical embedding $X^* \to X^{***}$ w^*-to-w^* continuous?

5. Prove that the closed unit ball B_X of a normed space X is weakly closed. Use this to show that if every norm-Cauchy sequence in X is weakly convergent then X is a Banach space.

6. Show that in ℓ_∞ the set $\{e_n : n \in \mathbb{N}\} \cup \{0\}$ is weakly compact but not norm compact.

7. Let $x_n \xrightarrow{w} 0$ in a Banach space X. Show that for all $\varepsilon > 0$ and for all $m \in \mathbb{N}$ there exists $n > m$ such that for all $x^* \in B_{X^*}$ there exists $i \in \mathbb{N}$ such that $m < i < n$ and $|x^*(x_i)| < \varepsilon$.

8. Let Z be a subspace of X^* that separates the points of X. Show that if B_X is $\sigma(X, Z)$-compact, then X is a dual space. More precisely, $X \cong Z^*$.

9. Given $F \subset X^*$ of $\dim F < \infty$, $\varphi \in B_{X^{**}}$ and $\varepsilon > 0$, show that there exists $x \in X$ with $\|x\| < 1 + \varepsilon$ such that $\hat{x}$ and φ agree on F. Use this to give an alternative proof of Goldstine's theorem.

10. Show that every weakly compact subset of ℓ_∞ is norm separable.

11. Let X be a Banach space. Prove directly, without the characterization of reflexivity in terms of w-compactness of the unit ball, that X is reflexive if and only if X^* is reflexive. Show that if Y is a closed subspace of X, then X is reflexive if and only if Y and X/Y are reflexive.

12. Show that none of the spaces c_0, ℓ_1, ℓ_∞, $L_1[0, 1]$ and $L_\infty[0, 1]$ is reflexive.

13. Let Y be a closed subspace of a Banach space X and $q\colon X \to X/Y$ be the quotient map. Show that q is an open map with respect to the weak topologies of X and X/Y.