1. Let X be a normed space. Show that for $f \in S_{X^*}$ we have $|f(x)| = \operatorname{dist}(x, \ker f)$ for all $x \in X$. Show further that for a closed subspace Y of X and $x_0 \notin Y$ there is $f \in S_{X^*}$ with $Y \subset \ker f$ and $f(x_0) = d(x_0, Y)$.

2. Prove Riesz's lemma: if Y is a proper, closed subspace of a normed space X, then for all $\varepsilon > 0$ there exists $x \in S_X$ with $\operatorname{dist}(x, Y) = \inf\{\|x - y\| : y \in Y\} > 1 - \varepsilon$.

3. Let Y be a closed subspace of a normed space X. Show that the topology on X/Y induced by the quotient norm is the quotient topology induced by the quotient map $q \colon X \to X/Y$. Show further that Y and X/Y are complete if and only if X is complete.

4. Show that every separable Banach space X is the quotient of ℓ_1, *i.e.*, that there is a closed subspace Y of ℓ_1 with $X \cong \ell_1/Y$.

5. Let Y be a closed subspace of a normed space X and let $Y^\perp = \{f \in X^* : f\restriction_Y = 0\}$. Show that $\big(X/Y\big)^* \cong Y^\perp$ and that $Y^* \cong X^*/Y^\perp$.

6. Show that a Banach space X with its weak topology is a completely regular topological space: given a weakly closed subset C of X and $p \notin C$, there is a continuous function $f \colon (X, w) \to [0, 1]$ such that $f(p) = 1$ and $f = 0$ on C.

7. Show the following quantitative version of Lemma 2.3. Let X be a normed space, $f, g_1, g_2, \dots, g_n \in X^*$ and $\varepsilon > 0$. Assume that the restriction of f to $\bigcap_{i=1}^n \ker g_i$ has norm at most ε. Deduce that $d\big(f, \operatorname{span}\{g_1, \dots, g_n\}\big) \leqslant \varepsilon$.

8. Show that the weak and norm topologies on a normed space X coincide if and only if $\dim X < \infty$. Show that the weak topology of an infinite-dimensional normed space and the w^*-topology of the dual space of an infinite-dimensional Banach space are not metrizable.

9. Let $T \colon X \to Y$ be a bounded linear map between Banach spaces. Show that
(i) T is an into isomorphism if and only if T^* is onto;
(ii) T^* is an into isomorphism if and only if T is onto;
(iii) T^* is injective if and only if $T(X)$ is dense in Y;
(iv) $T(X)$ is closed if and only if $T^*(Y^*)$ is closed.

10. For a subset A of a normed space X, we define the *annihilator of* A as the subset $A^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in A\}$ of X^*. Similarly, for $B \subset X^*$, we define the *preannihilator of* B as the subset $B_\perp = \{x \in X : f(x) = 0 \text{ for all } f \in B\}$ of X.

(i) Show that $\overline{\operatorname{span}} A = \left(A^\perp\right)_\perp$ for any $A \subset X$.

(ii) Show that $\overline{\operatorname{span}}^{w^*} B = \left(B_\perp\right)^\perp$ for any $B \subset X^*$. Deduce that a w^*-closed subspace Y of X^* is a dual space: there is a normed space Z such that $Y \cong Z^*$.

(iii) Prove the following for a bounded linear map T between normed spaces.

(a) $\ker T = (\operatorname{im} T^*)_\perp$ and $\ker T^* = (\operatorname{im} T)^\perp$.

(b) $\overline{\operatorname{im} T} = (\ker T^*)_\perp$ and $\overline{\operatorname{im} T^*}^{w^*} = (\ker T)^\perp$.

11. Let Ω be a set and $\mathcal{F}$ be a σ-field on Ω. Prove carefully that the set $L_\infty(\Omega, \mathcal{F})$ of all bounded, measurable, scalar-valued functions on Ω is a Banach space in the supremum norm: $\|f\|_\infty = \sup_{\omega\in\Omega} |f(\omega)|$. The aim of this question is to identify $L_\infty(\Omega, \mathcal{F})^*$.

A *finitely additive measure on* $\mathcal{F}$ is a (real or complex) function ν on $\mathcal{F}$ such that $\nu(\emptyset) = 0$ and $\nu(A \cup B) = \nu(A) + \nu(B)$ whenever $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$. The *total variation measure* $|\nu|$ *of* ν is defined as follows.

$$|\nu|(A) = \sup\left\{ \sum_{k=1}^{n} |\nu(A_k)| : A = \bigcup_{k=1}^{n} A_k \text{ is a measurable partition of } A \right\}.$$

The *total variation of* ν is $\|\nu\|_1 = |\nu|(\Omega)$. We say ν is *bounded* if $\|\nu\|_1 < \infty$. Show that the space $\operatorname{ba}(\Omega, \mathcal{F})$ of all bounded, finitely additive measures on $\mathcal{F}$ is a Banach space in the total variation norm and that it is isometrically isomorphic to $L_\infty(\Omega, \mathcal{F})^*$.

12. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that $L_\infty(\mu)$ is a quotient of $L_\infty(\Omega, \mathcal{F})$. Deduce that $L_\infty(\mu)^*$ is a subspace of $\operatorname{ba}(\Omega, \mathcal{F})$ and identify that subspace.

13. Let $0 < p < 1$ and define $L_p(0, 1)$ to be the space of Lebesgue measurable functions $f \colon [0, 1] \to \mathbb{R}$ for which $\int_0^1 |f(x)|^p \,\mathrm{d}x < \infty$. Show that

$$d(f, g) = \int_0^1 |f(x) - g(x)|^p \,\mathrm{d}x$$

defines a metric on $L_p(0, 1)$ (as usual, we identify functions that are equal a.e.). Identify the dual space of $L_p(0, 1)$, *i.e.*, the space of linear functionals on $L_p(0, 1)$ that are continuous with respect to d.