Part III Functional Analysis

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Contents

1 The Hahn–Banach extension theorems

Dual space. Let X be a normed space. The *dual space* X^* *of* X is the space of all bounded linear functionals on X. The dual space is a Banach space in the operator norm which is defined for $f \in X^*$ as

$$\|f\| = \sup \big\{ |f(x)| : x \in B_X \big\} .$$

Recall that $B_X = \{x \in X : \|x\| \leqslant 1\}$ is the closed unit ball of X and that $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of X.

Examples. $\ell_p^* \cong \ell_q$ for $1 \leqslant p < \infty$, $1 < q \leqslant \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We also have $\mathrm{c}_0^* \cong \ell_1$. If H is a Hilbert space then $H^* \cong H$ (conjugate-linear in the complex case).

Notation. 1. Given normed spaces X and Y, we write $X \sim Y$ if X and Y are isomorphic, *i.e.*, when there is a linear bijection $T\colon X \to Y$ such that both T and T^{-1} are continuous. (Recall that by the Open Mapping Theorem, if X and Y are both complete and T is a continuous linear bijection, then T^{-1} is automatically continuous.)

2. Given normed spaces X and Y, we write $X \cong Y$ if X and Y are isometrically isomorphic, *i.e.*, when there is a surjective linear map $T\colon X \to Y$ such that $\|Tx\| = \|x\|$ for all $x \in X$. It follows that T is a continuous linear bijection and that T^{-1} is also isometric, and hence continuous.
3. For $x \in X$ and $f \in X^*$, we shall sometimes denote $f(x)$, the action of f on x, by $\langle x, f\rangle$. By definition of the operator norm we have

$$|\langle x, f\rangle| = |f(x)| \leqslant \|f\| \cdot \|x\| \ .$$

When X is a Hilbert space and we identify X^* with X in the usual way, then $\langle x, f\rangle$ is the inner product of x and f.

Definition. Let X be a real vector space. A functional $p\colon X \to \mathbb{R}$ is called

- *positive homogeneous* if $p(tx) = tp(x)$ for all $x \in X$ and $t \geqslant 0$, and
- *subadditive* if $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$.

Theorem 1. (Hahn–Banach) Let X be a real vector space and p be a positive homogeneous, subadditive functional on X. Let Y be a subspace of X and $g\colon Y \to \mathbb{R}$ be a linear functional such that $g(y) \leqslant p(y)$ for all $y \in Y$. Then there is a linear functional $f\colon X \to \mathbb{R}$ such that $f\restriction_Y = g$ and $f(x) \leqslant p(x)$ for all $x \in X$.

Zorn's Lemma. This is used in the proof, so we recall it here. Let $(P, \leqslant)$ be a poset (partially ordered set). An element $x \in P$ is an *upper bound* for a subset A of P if $a \leqslant x$ for all $a \in A$. A subset C of P is called a *chain* if it is linearly ordered by $\leqslant$. An element $x \in P$ is a *maximal element* of P if whenever $x \leqslant y$ for some $y \in P$, we have $y = x$. Zorn's Lemma states that if $P \neq \emptyset$ and every non-empty chain in P has an upper bound, then P has a maximal element.

Definition. A *seminorm* on a real or complex vector space X is a functional $p\colon X \to \mathbb{R}$ such that

- $p(x) \geqslant 0$ for all $x \in X$;
- $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and for all scalars λ;
- $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$.

Note. "norm" $\implies$ "seminorm" $\implies$ "positive homogeneous+subadditive".

Theorem 2. (Hahn–Banach) Let X be a real or complex vector space and p be a seminorm on X. Let Y be a subspace of X and g be a linear functional on Y such that $|g(y)| \leqslant p(y)$ for all $y \in Y$. Then there is a linear functional f on X such that $f\restriction_Y = g$ and $|f(x)| \leqslant p(x)$ for all $x \in X$.

Remark. It follows from the proof given in lectures that for a complex normed space X, the map $f \mapsto \mathrm{Re}(f)\colon \big(X^*\big)_{\mathbb{R}} \to \big(X_{\mathbb{R}}\big)^*$ is an isometric isomorphism. Here $Y_{\mathbb{R}}$, for a complex vector space Y, denotes the real vector space obtained from Y by restricting scalar multiplication to the reals.

Corollary 3. Let X be a real or complex vector space and p be a seminorm on X. For every $x_0 \in X$ there is a linear functional f on X with $f(x_0) = p(x_0)$ and $|f(x)| \leqslant p(x)$ for all $x \in X$.

Theorem 4. (Hahn–Banach) Let X be a real or complex normed space. Then

(i) Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ with $f\restriction_Y = g$ and $\|f\| = \|g\|$.

(ii) Given a non-zero $x_0 \in X$, there exists $f \in S_{X^*}$ with $f(x_0) = \|x_0\|$.

Remarks. 1. Part (i) can be viewed as a linear version of Tietze's extension theorem. The latter states that if L is a closed subset of a compact Hausdorff space K, and $g\colon L \to \mathbb{R}$ (or $\mathbb{C}$) is continuous, then there is a continuous function $f\colon K \to \mathbb{R}$ (respectively, $\mathbb{C}$) such that $f\restriction_L = g$ and $\|f\|_\infty = \|g\|_\infty$.
2. Part (ii) implies that X^* separates the points of X: given $x \neq y$ in X, there exists $f \in X^*$ such that $f(x) \neq f(y)$. Thus there are "plenty" bounded linear functionals on X.
3. The element $f \in X^*$ in part (ii) is called a *norming functional for* x_0. It shows that

$$\|x_0\| = \max\big\{|\langle x_0, g\rangle| : g \in B_{X^*}\big\} .$$

Another name for f is *support functional at* x_0. Assume X is a real normed space and that $\|x_0\| = 1$. Then $B_X \subset \{x \in X : f(x) \leqslant 1\}$, and hence the hyperplane $\{x \in X : f(x) = 1\}$ can be thought of as a tangent to B_X at x_0.

Bidual. Let X be a normed space. Then $X^{**} = \big(X^*\big)^*$ is called the *bidual* or *second dual of* X. It is the Banach space of all bounded linear functionals on X^* with the operator norm. For $x \in X$ we define $\hat{x}\colon X^* \to \mathbb{R}$ (or $\mathbb{C}$) by $\hat{x}(f) = f(x)$ (*evaluation at* x). Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leqslant \|x\|$. The map $x \mapsto \hat{x}\colon X \to X^{**}$ is called the *canonical embedding*.

Theorem 5. The canonical embedding defined above is an isometric isomorphism of X *into* X^{**}.

Remarks. 1. Using the bracket notation, we have

$$\langle f, \hat{x}\rangle = \langle x, f\rangle = f(x) \qquad x \in X,\ f \in X^* .$$

2. The image $\widehat{X} = \{\hat{x} : x \in X\}$ of the canonical embedding is closed in X^{**} if and only if X is complete.
3. In general, the closure of $\widehat{X}$ in X^{**} is a Banach space which contains an isometric copy of X as a dense subspace. We have thus proved that every normed space has a completion.

Definition. X is *reflexive* if the canonical embedding of X into X^{**} is surjective.

Examples. 1. The spaces ℓ_p, $1 < p < \infty$, Hilbert spaces, finite-dimensional spaces are reflexive. Later we prove that the spaces $L_p(\mu)$, $1 < p < \infty$, are also reflexive.
2. The spaces c_0, ℓ_1, $L_1[0,1]$ are not reflexive.

Remark. Every reflexive space X is complete with $X \cong X^{**}$. There are Banach spaces X with $X \cong X^{**}$ which are not reflexive. So for $1 < p < \infty$, it is not sufficient to say that $\ell_p^{**} \cong \ell_q^* \cong \ell_p$ (where q is the conjugate index of p) to deduce that ℓ_p is reflexive. One also has to verify that this isometric isomorphism is the canonical embedding.

Dual operators. Recall that for normed spaces X, Y we denote by $\mathcal{B}(X, Y)$ the space of all bounded linear maps $T\colon X \to Y$. This is a normed space in the operator norm:

$$\|T\| = \sup\big\{\|Tx\| : x \in B_X\big\} .$$

Moreover, $\mathcal{B}(X, Y)$ is complete if Y is.

For $T \in \mathcal{B}(X, Y)$, the *dual operator* of T is the map $T^*\colon Y^* \to X^*$ defined by $T^*(g) = g \circ T$ for $g \in Y^*$. In bracket notation:

$$\langle x, T^* g\rangle = \langle Tx, g\rangle \qquad x \in X,\ g \in Y^* .$$

T^* is linear and bounded with $\|T^*\| = \|T\|$ (uses Theorem 4(ii)).

Remark. When X and Y are Hilbert spaces, the dual operator T^* corresponds to the adjoint of T after identifying X^* and Y^* with X and Y, respectively.

Example. Let $1 < p < \infty$ and consider the right shift $R\colon \ell_p \to \ell_p$. Then $R^*\colon \ell_q \to \ell_q$ (q the conjugate index of p) is the left shift.

Properties of dual operators. 1. $\big(\mathrm{Id}_X\big)^* = \mathrm{Id}_{X^*}$
2. $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ for scalars λ, μ and for $S, T \in \mathcal{B}(X, Y)$. Note that unlike for adjoints there is no complex conjugation here. That is because the identification of a Hilbert space with its dual is conjugate linear in the complex case.
3. $(ST)^* = T^* S^*$ for $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$.
4. The map $T \mapsto T^*\colon \mathcal{B}(X, Y) \to \mathcal{B}(Y^*, X^*)$ is an *into* isometric isomorphism.
5. The following diagram

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ \downarrow & & \downarrow \\ X^{**} & \xrightarrow{\ T^{**}\ } & Y^{**} \end{array}$$

commutes, where the vertical arrows represent canonical embeddings.

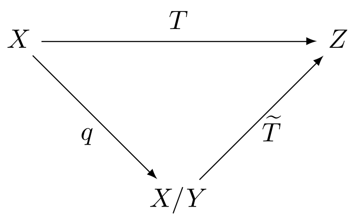
Remark. It follows from the above properties that if $X \sim Y$ then $X^* \sim Y^*$.

Quotient space. Let X be a normed space and Y a closed subspace. Then the quotient space X/Y becomes a normed space in the *quotient norm* defined as follows.

$$\|x + Y\| = \inf\big\{\|x + y\| : y \in Y\big\} \qquad x \in X .$$

The quotient map $q\colon X \to X/Y$ is a surjective linear map that is bounded with $\|q\| \leqslant 1$. It maps the open unit ball $D_X = \{x \in X : \|x\| < 1\}$ onto $D_{X/Y}$. It follows that $\|q\| = 1$ (if $X \neq Y$) and that q is an open map (it maps open sets onto open sets).

If $T\colon X \to Z$ is a bounded linear map with $Y \subset \ker(T)$, then there is a unique map $\widetilde{T}\colon X/Y \to Z$ such that the diagram



commutes, and moreover $\widetilde{T}$ is linear, bounded and $\|\widetilde{T}\| = \|T\|$.

Theorem 6. Let X be a normed space. If X^* is separable, then so is X.

Remark. The converse is false. *E.g.,* $X = \ell_1$ is separable but $X^* \cong \ell_\infty$ is not.

Theorem 7. Every separable normed space is isometrically isomorphic to a subspace of ℓ_∞.

Remarks. 1. The result says that ℓ_∞ is *isometrically universal* for the class $\mathcal{SB}$ of all separable Banach spaces. We will later see that there is a separable space with the same property.
2. A dual version of the above result states that every separable Banach spaces is a quotient of ℓ_1.

Theorem 8. (Vector-valued Liouville) Let X be a complex Banach space and $f\colon \mathbb{C} \to X$ be a bounded, holomorphic function. Then f is constant.

Locally Convex Spaces. A *locally convex space (LCS)* is a pair $(X, \mathcal{P})$, where X is a real or complex vector space and $\mathcal{P}$ is a family of seminorms on X that separates the points of X in the sense that for every $x \in X$ with $x \neq 0$, there is a seminorm $p \in \mathcal{P}$ with $p(x) \neq 0$. The family $\mathcal{P}$ defines a topology on X: a set $U \subset X$ is *open* if and only if for all $x \in U$ there exist $n \in \mathbb{N}$, $p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon > 0$ such that

$$\big\{ y \in X : \, p_k(y - x) < \varepsilon \text{ for } 1 \leqslant k \leqslant n \big\} \subset U \ .$$

Remarks. 1. Addition and scalar multiplication are continuous.
2. The topology of X is Hausdorff as $\mathcal{P}$ separates the points of X.
3. A sequence $x_n \to x$ in X if and only if $p(x_n - x) \to 0$ for all $p \in \mathcal{P}$. (The same holds for nets.)
4. For a subspace Y of X define $\mathcal{P}_Y = \{p\restriction_Y \colon p \in \mathcal{P}\}$. Then $(Y, \mathcal{P}_Y)$ is a locally convex space, and the corresponding locally convex topology on Y is nothing else but the subspace topology on Y induced by X.
5. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of seminorms on X both of which separate the points of X. We say $\mathcal{P}$ and $\mathcal{Q}$ are *equivalent* and write $\mathcal{P} \sim \mathcal{Q}$ if they define the same topology on X.
The topology of a locally convex space $(X, \mathcal{P})$ is metrizable if and only if there is a countable $\mathcal{Q}$ with $\mathcal{Q} \sim \mathcal{P}$.

Definition. A *Fréchet space* is a complete metrizable locally convex space.

Examples. 1. Every normed space $(X, \|\cdot\|)$ is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.
2. Let U be a non-empty, open subset of $\mathbb{C}$, and let $\mathcal{O}(U)$ denote the space of all holomorphic functions $f : U \to \mathbb{C}$. For a compact set $K \subset U$ and for $f \in \mathcal{O}(U)$ set $p_K(f) = \sup_{z\in K}|f(z)|$. Set $\mathcal{P} = \{p_K : K \subset U, K \text{ compact}\}$. Then $(\mathcal{O}(U), \mathcal{P})$ is a locally convex space whose topology is the topology of local uniform convergence.

There exist compact sets $K_n \subset U$, $n \in \mathbb{N}$, such that $K_n \subset \mathrm{int}(K_{n+1})$ and $U = \bigcup_n K_n$. Then $\{p_{K_n} : n \in \mathbb{N}\}$ is countable and equivalent to $\mathcal{P}$. Hence $(\mathcal{O}(U), \mathcal{P})$ is metrizable and in fact it is a Fréchet space.

The topology of local uniform convergence is not *normable*: the topology is not induced by a norm. This follows, for example, from Montel's theorem: given a sequence (f_n) in $\mathcal{O}(U)$ such that $(f_n\restriction_K)$ is bounded in $(C(K), \|\cdot\|_\infty)$ for every compact $K \subset U$, there is a subsequence of (f_n) that converges locally uniformly.
3. Fix $d \in \mathbb{N}$ and a non-empty open set $\Omega \subset \mathbb{R}^d$. Let $C^\infty(\Omega)$ denote the space of all infinitely differentiable functions $f : \Omega \to \mathbb{R}$. Every *multi-index* $\alpha = (\alpha_1, \dots, \alpha_d) \in \left(\mathbb{Z}_{\geqslant 0}\right)^d$ gives rise to a partial differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} .$$

For a multi-index α, a compact set $K \subset \Omega$, and $f \in \mathcal{O}(U)$ define

$$p_{K,\alpha}(f) = \sup\left\{|(D^\alpha f)(x)| : x \in K\right\} .$$

Set $\mathcal{P} = \{p_{K,\alpha} : K \subset \Omega \text{ compact}, \alpha \in (\mathbb{Z}_{\geqslant 0})^d\}$. Then $(C^\infty(\Omega), \mathcal{P})$ is a locally convex space. It is a Fréchet space and is not normable.

Lemma 9. Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be locally convex spaces. Let $T : X \to Y$ be a linear map. Then TFAE:

(i) T is continuous.

(ii) T is continuous at 0.

(iii) For all $q \in Q$ there exist $n \in \mathbb{N}$, $p_1, \dots, p_n \in \mathcal{P}$ and $C \geqslant 0$ such that

$$q(Tx) \leqslant C \max_{1\leqslant k\leqslant n} p_k(x) \qquad \text{for all } x \in X .$$

Dual space. Let $(X, \mathcal{P})$ be a locally convex space. The *dual space* X^* of X is the space of all linear functionals on X that are continuous with respect to the locally convex topology of X.

Lemma 10. Let f be a linear functional on a locally convex space X. Then $f \in X^*$ if and only if $\ker f$ is closed.

Theorem 11. (Hahn–Banach) Let $(X, \mathcal{P})$ be a locally convex space.

(i) Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ with $f\restriction_Y = g$.

(ii) Given a closed subspace Y of X and $x_0 \in X \setminus Y$, there exists $f \in X^*$ such that $f\restriction_Y = 0$ and $f(x_0) \neq 0$.

Remark. It follows that X^* separates the points of X.

The dual space of L_p. We fix a measure space $(\Omega, \mathcal{F}, \mu)$. Let $1 \leqslant p < \infty$ and let q be the conjugate index of p (*i.e.*, $1 < q \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$). For $g \in L_q = L_q(\mu)$ define

$$\varphi_g \colon L_p \to \text{scalars} , \qquad \varphi_g(f) = \int_\Omega fg \, \mathrm{d}\mu .$$

By Hölder's inequality we have φ_g is well-defined and $|\varphi_g(f)| \leqslant \|f\|_p \cdot \|g\|_q$. By linearity of integration, φ_g is linear, and hence $\varphi_g \in L_p^*$ with $\|\varphi_g\| \leqslant \|g\|_q$. We have thus obtained a function

$$\varphi \colon L_q \to L_p^* , \quad g \mapsto \varphi_g .$$

This is linear by linearity of integration and bounded with $\|\varphi\| \leqslant 1$.

Theorem 12. Let $(\Omega, \mathcal{F}, \mu)$, p, q, φ be as above.

(i) If $1 < p < \infty$, then φ is an isometric isomorphism. Thus $L_p^* \cong L_q$.

(ii) If $p = 1$ and in addition μ is σ-finite, then φ is an isometric isomorphism. Thus, in this case, we have $L_1^* \cong L_\infty$.

Remarks. 1. Recall that μ is *σ-finite* if there is a sequence (A_n) in $\mathcal{F}$ such that $\Omega = \bigcup A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.
2. One approach for proving surjectivity of φ is via the Radon–Nikodym theorem. We shall follow a different path via uniform convexity in Chapter 4.

Complex measures. Let Ω be a set and $\mathcal{F}$ be a σ-field on Ω. A *complex measure on* $\mathcal{F}$ is a countably additive set function $\nu \colon \mathcal{F} \to \mathbb{C}$. The *total variation measure* $|\nu|$ of ν is defined at $A \in \mathcal{F}$ as follows.

$$|\nu|(A) = \sup \Big\{ \sum_{k=1}^{n} |\nu(A_k)| : A = \bigcup_{k=1}^{n} A_k \text{ is a measurable partition of } A \Big\} .$$

(*Measurable partition* means that $A_k \in \mathcal{F}$ for all k, and $A_j \cap A_k = \emptyset$ for all $j \neq k$.) It is easy to check that $|\nu| \colon \mathcal{F} \to [0, \infty]$ is a positive measure. (The expression *positive measure* simply means measure but is used for emphasis to distinguish it from complex and signed measures.) It is also straightforward to verify that $|\nu|$ is the smallest positive measure that dominates ν. It is in fact a finite measure, *i.e.*, that $|\nu|(\Omega) < \infty$. We define the *total variation* $\|\nu\|_1$ *of* ν by $\|\nu\|_1 = |\nu|(\Omega)$.

Signed measures. Let $(\Omega, \mathcal{F})$ be as before. A *signed measure on* $\mathcal{F}$ is a countably additive set function $\nu \colon \mathcal{F} \to \mathbb{R}$. Every signed measure is in particular a complex measure.

Theorem 13. Let $\nu \colon \mathcal{F} \to \mathbb{R}$ be a signed measure on the σ-field $\mathcal{F}$. Then there exist unique finite positive measures ν^+ and ν^- satisfying $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$.

Remarks. 1. This decomposition $\nu = \nu^+ - \nu^-$ of ν is called the *Jordan decomposition* of ν.

2. Let $\nu\colon \mathcal{F} \to \mathbb{C}$ be a complex measure. Then $\mathrm{Re}(\nu)$ and $\mathrm{Im}(\nu)$ are signed measures, so they have Jordan decompositions $\nu_1 - \nu_2$ and $\nu_3 - \nu_4$, respectively. We then obtain the expression $\nu = \nu_1 - \nu_2 + \mathrm{i}\nu_3 - \mathrm{i}\nu_4$ called the Jordan decomposition of ν. It follows that $\nu_k \leqslant |\nu|$ for all k, and $|\nu| \leqslant \nu_1 + \nu_2 + \nu_3 + \nu_4$. From this we deduce that $|\nu|$ is a finite measure.

Integration with respect to complex measures. Let Ω be a set, $\mathcal{F}$ be a σ-field on Ω, and ν be a complex measure on $\mathcal{F}$. A measurable function $f\colon \Omega \to \mathbb{C}$ is *ν-integrable* if $\int_\Omega |f|\,\mathrm{d}|\nu| < \infty$. In that case we define

$$\int_\Omega f\,\mathrm{d}\nu = \int_\Omega f\,\mathrm{d}\nu_1 - \int_\Omega f\,\mathrm{d}\nu_2 + \mathrm{i}\int_\Omega f\,\mathrm{d}\nu_3 - \mathrm{i}\int_\Omega f\,\mathrm{d}\nu_4\ ,$$

where $\nu = \nu_1 - \nu_2 + \mathrm{i}\nu_3 - \mathrm{i}\nu_4$ is the Jordan decomposition of ν. Note that f is ν-integrable if and only if f is ν_k-integrable for each k. The following properties are easy to check:

1. $\int_\Omega \mathbf{1}_A\,\mathrm{d}\nu = \nu(A)$ for all $A \in \mathcal{F}$.

2. Linearity: given ν-integrable functions f, g and complex numbers a, b, the function $af + bg$ is ν-integrable, and $\int_\Omega (af + bg)\,\mathrm{d}\nu = a\int_\Omega f\,\mathrm{d}\nu + b\int_\Omega g\,\mathrm{d}\nu$.

3. Dominated convergence (D.C.): let f_n, $n \in \mathbb{N}$, be measurable functions that converge a.e. to a measurable function f. Assume that there exists $g \in L_1(|\nu|)$ with $|f_n| \leqslant g$ for all n. Then f_n, f are ν-integrable and $\int_\Omega f_n\,\mathrm{d}\nu \to \int_\Omega f\,\mathrm{d}\nu$.

4. $\left|\int_\Omega f\,\mathrm{d}\nu\right| \leqslant \int_\Omega |f|\,\mathrm{d}|\nu|$ for all ν-integrable f.

***C(K)* spaces.** We fix a compact Hausdorff space K. We shall be interested in the following spaces and sets.

$$C(K) = \{f\colon K \to \mathbb{C} : f \text{ is continuous}\}$$

$$C^{\mathbb{R}}(K) = \{f\colon K \to \mathbb{R} : f \text{ is continuous}\}$$

$$C^{+}(K) = \{f \in C(K) : f \geqslant 0 \text{ on } K\}$$

$$M(K) = C(K)^* = \{\varphi\colon C(K) \to \mathbb{C} : \varphi \text{ is linear and continuous}\}$$

$$M^{\mathbb{R}}(K) = \{\varphi \in M(K) : \varphi(f) \in \mathbb{R} \text{ for all } f \in C^{\mathbb{R}}(K)\}$$

$$M^{+}(K) = \{\varphi\colon C(K) \to \mathbb{C} : \varphi \text{ is linear and } \varphi(f) \geqslant 0 \text{ for all } f \in C^{+}(K)\}$$

Note. 1. $M^{\mathbb{R}}(K)$ is a closed, real-linear subspace of $M(K)$. $C^{\mathbb{R}}(K)^* \cong M^{\mathbb{R}}(K)$. $f \mapsto f\restriction_{C^{\mathbb{R}}(K)}\colon M^{\mathbb{R}}(K) \to C^{\mathbb{R}}(K)^*$ is an isometric real-linear isometry.

2. Elements of $M^{+}(K)$ are called *positive linear functionals*. These are automatically continuous, and in fact

$$M^{+}(K) = \{\varphi \in M(K) : \|\varphi\| = \varphi(\mathbf{1}_K)\}\ .$$

Borel measures and regularity. Let X be a Hausdorff topological space and $\mathcal{G}$ be the collection of open subsets of X. The *Borel σ-field on X* is defined to be $\mathcal{B} = \sigma(\mathcal{G})$, the σ-field generated by $\mathcal{G}$, *i.e.*, the smallest σ-field on X containing $\mathcal{G}$. Equivalently, $\mathcal{B}$ is the intersection of all σ-fields on X that contain $\mathcal{G}$. Members of $\mathcal{B}$ are called *Borel sets*.

A *Borel measure on* X is a measure on $\mathcal{B}$. Given a Borel measure μ on X, we say μ is *regular* if the following hold:

(i) $\mu(E) < \infty$ for all compact $E \subset X$;

(ii) $\mu(A) = \inf\{\mu(U) : A \subset U \in \mathcal{G}\}$ for all $A \in \mathcal{B}$;

(iii) $\mu(U) = \sup\{\mu(E) : E \subset U, \ E \text{ compact}\}$ for all $U \in \mathcal{G}$.

A complex Borel measure ν on X is defined to be *regular* if $|\nu|$ is a regular measure on X.

Note that if X is compact Hausdorff, then a Borel measure μ on X is regular if and only if

$$\mu(X) < \infty \quad \text{and} \quad \mu(A) = \inf\{\mu(U) : A \subset U \in \mathcal{G}\} \quad \forall\, A \in \mathcal{B}$$

which in turn is equivalent to

$$\mu(X) < \infty \quad \text{and} \quad \mu(A) = \sup\{\mu(E) : E \subset A, \ E \text{ closed}\} \quad \forall\, A \in \mathcal{B} \ .$$

Example. Lebesgue measure on $\mathbb{R}$ is a regular Borel measure.

The dual space of $C(K)$. Let ν be a complex Borel measure on K. For $f \in C(K)$ we have

$$\int_K |f| \,\mathrm{d}|\nu| \leqslant \|f\|_\infty \cdot |\nu|(K) \ ,$$

and hence f is ν-integrable. The function $\varphi \colon C(K) \to \mathbb{C}$ given by $\varphi(f) = \int_K f \,\mathrm{d}\nu$ is linear and bounded with $\|\varphi\| \leqslant \|\nu\|_1$. Thus $\varphi \in M(K)$. Note that if ν is a signed measure, then $\varphi \in M^{\mathbb{R}}(K)$, and if ν is a positive measure, then $\varphi \in M^+(K)$.

Theorem 14. (Riesz Representation Theorem) For every $\varphi \in M(K)$ there is a unique regular complex Borel measure ν on K that represents φ:

$$\varphi(f) = \int_K f \,\mathrm{d}\nu \qquad \text{for all } f \in C(K) \ .$$

Moreover, we have $\|\varphi\| = \|\nu\|_1$. If $\varphi \in M^{\mathbb{R}}(K)$ then ν is a signed measure, and if $\varphi \in M^+(K)$, then ν is a positive measure.

Corollary 15. The space of regular complex Borel measures on K is a complex Banach space in the total variation norm, and it is isometrically isomorphic to $M(K) = C(K)^*$.

The space of regular signed Borel measures on K is a real Banach space in the total variation norm, and it is isometrically isomorphic to $M^{\mathbb{R}}(K) \cong C^{\mathbb{R}}(K)^*$.

2 Weak topologies

Let X be a set and $\mathcal{F}$ be a family of functions such that each $f \in \mathcal{F}$ is a function $f \colon X \to Y_f$ where Y_f is a topological space. The *weak topology* $\sigma(X, \mathcal{F})$ *of* X is the smallest topology on X with respect to which every $f \in \mathcal{F}$ is continuous.

Remarks. 1. The family $\mathcal{S} = \{f^{-1}(U) : f \in \mathcal{F},\ U \subset Y_f \text{ an open set}\}$ is a sub-base for $\sigma(X, \mathcal{F})$. This means that $\sigma(X, \mathcal{F})$ is the topology generated by $\mathcal{S}$, *i.e.*, the smallest topology containing $\mathcal{S}$. Equivalently, the family of finite intersections of members of $\mathcal{S}$ is a base for $\sigma(X, \mathcal{F})$, *i.e.*, $\sigma(X, \mathcal{F})$ consists of arbitrary unions of finite intersections of members of $\mathcal{S}$. To spell this out, a set $V \subset X$ is open in the weak topology $\sigma(X, \mathcal{F})$ if and only if for all $x \in V$ there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in \mathcal{F}$ and open sets $U_j \subset Y_{f_j}$, $1 \leqslant j \leqslant n$, such that

$$x \in \bigcap_{j=1}^{n} f_j^{-1}(U_j) \subset V .$$

To put it in another way, $V \subset X$ is open in the weak topology if and only if for all $x \in V$ there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in \mathcal{F}$ and open neighbourhoods U_j of $f_j(x)$ in Y_{f_j}, $1 \leqslant j \leqslant n$, such that $\bigcap_{j=1}^{n} f_j^{-1}(U_j) \subset V$.
2. Suppose that $\mathcal{S}_f$ is a sub-base for the topology of Y_f for each $f \in \mathcal{F}$. Then $\{f^{-1}(U) : f \in \mathcal{F},\ U \in \mathcal{S}_f\}$ is a sub-base for $\sigma(X, \mathcal{F})$.
3. If Y_f is a Hausdorff space for each $f \in \mathcal{F}$ and $\mathcal{F}$ separates the points of X (for all $x \neq y$ in X there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$), then $\sigma(X, \mathcal{F})$ is Hausdorff.
4. Let $Y \subset X$ and set $\mathcal{F}_Y = \{f\restriction_Y : f \in \mathcal{F}\}$. Then $\sigma(X, \mathcal{F})\restriction_Y = \sigma(Y, \mathcal{F}_Y)$, *i.e.*, the subspace topology on Y induced by the weak topology $\sigma(X, \mathcal{F})$ of X is the same as the weak topology on Y defined by $\mathcal{F}_Y$.
5. (Universal property) Let Z be a topological space and $g\colon Z \to X$ a function. Then g is continuous if and only if $f \circ g\colon Z \to Y_f$ is continuous for every $f \in \mathcal{F}$. This universal property characterizes the weak topology (*cf.* Examples Sheet 2).

Examples. 1. Let X be a topological space, $Y \subset X$ and $\iota\colon Y \to X$ be the inclusion map. Then the weak topology $\sigma(Y, \{\iota\})$ is the subspace topology of Y induced by X.
2. Let Γ be an arbitrary set, and let X_γ be a topological space for each $\gamma \in \Gamma$. Let X be the Cartesian product $\prod_{\gamma \in \Gamma} X_\gamma$. Thus

$$X = \{x : x \text{ is a function with domain } \Gamma, \text{ and } x(\gamma) \in X_\gamma \text{ for all } \gamma \in \Gamma\} .$$

For $x \in X$ we often write x_γ instead of $x(\gamma)$, and think of x as the "Γ-tuple" $(x_\gamma)_{\gamma \in \Gamma}$. For each γ we consider the function $\pi_\gamma\colon X \to X_\gamma$ given by $x \mapsto x(\gamma)$ (or $(x_\delta)_{\delta \in \Gamma} \mapsto x_\gamma$) called *evaluation at* γ or *projection onto* X_γ. The *product topology on* X is the weak topology $\sigma(X, \{\pi_\gamma : \gamma \in \Gamma\})$. Note that $V \subset X$ is open if and only if for all $x = (x_\gamma)_{\gamma \in \Gamma} \in V$ there exist $n \in \mathbb{N}$, $\gamma_1, \dots, \gamma_n \in \Gamma$ and open neighbourhoods U_j of x_{γ_j} in X_{γ_j} such that

$$\big\{y = (y_\gamma)_{\gamma \in \Gamma} : y_{\gamma_j} \in U_j \text{ for } 1 \leqslant j \leqslant n\big\} \subset V .$$

Proposition 1. Assume that X is a set, and for each $n \in \mathbb{N}$ we are given a metric space (Y_n, d_n) and a function $f_n\colon X \to Y_n$. Further assume that $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ separates the points of X. Then the weak topology $\sigma(X, \mathcal{F})$ of X is metrizable.

Remark. Without the assumption that $\mathcal{F}$ separates the points of X, we can conclude that $\sigma(X, \mathcal{F})$ is pseudo-metrizable.

Theorem 2. (Tychonov) The product of compact topological spaces is compact in the product topology.

Weak topologies on vector spaces. Let E be a real or complex vector space. Let F be a subspace of the space of all linear functionals on E that separates the points of E (for all $x \in E$, $x \neq 0$, there exists $f \in F$ such that $f(x) \neq 0$). Consider the weak topology $\sigma(E, F)$ on E. Note that $U \subset E$ is open if and only if for all $x \in U$ there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in F$ and $\varepsilon > 0$ such that

$$\{y \in E : |f_i(y - x)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \subset U \ .$$

For $f \in F$ define $p_f : E \to \mathbb{R}$ by $p_f(x) = |f(x)|$. Set $\mathcal{P} = \{p_f : f \in F\}$. Then $\mathcal{P}$ is a family of seminorms on E that separates the points of E, and the topology of the locally convex space $(E, \mathcal{P})$ is precisely $\sigma(E, F)$. In particular, $\sigma(E, F)$ is a Hausdorff topology with respect to which addition and scalar multiplication are continuous.

Lemma 3. Let E be as above. Let $f, g_1, g_2, \dots, g_n$ be linear functionals on E. If $\bigcap_{j=1}^{n} \ker g_j \subset \ker f$, then $f \in \operatorname{span}\{g_1, \dots, g_n\}$.

Proposition 4. Let E and F be as above. Then a linear functional f on E is $\sigma(E, F)$-continuous if and only if $f \in F$. In other words $(E, \sigma(E, F))^* = F$.

Note. Recall that $(E, \sigma(E, F))^*$ denotes the dual space of the locally convex space $(E, \sigma(E, F))$, *i.e.*, the space of linear functionals that are continuous with respect to $\sigma(E, F)$.

Examples. The following two examples of weak topologies on vector spaces are the central objects of interest in this chapter.

1. Let X be a normed space. The *weak topology on* X is the weak topology $\sigma(X, X^*)$ on X. Note that X^* separates the points of X by Hahn–Banach. We shall sometimes denote X with the weak topology by (X, w). Open sets in the weak topology are called *weakly open* or *w-open*. Note that $U \subset X$ is w-open if and only if for all $x \in U$ there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in X^*$ and $\varepsilon > 0$ such that

$$\{y \in X : |f_i(y - x)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \subset U \ .$$

2. Let X be a normed space. The *weak-star topology (or w^*-topology) on* X^* is the weak topology $\sigma(X^*, X)$ on X^*. Here we identify X with its image in X^{**} under the canonical embedding. We shall sometimes denote X^* with the w^*-topology by (X^*, w^*). Open sets in the weak-star topology are called *w^*-open*. Note that $U \subset X^*$ is w^*-open if and only if for all $f \in U$ there exist $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ such that

$$\{g \in X^* : |g(x_i) - f(x_i)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \subset U \ .$$

Properties. 1. (X, w) and (X^*, w^*) are locally convex spaces, so in particular they are Hausdorff, and addition and scalar multiplication are continuous.
2. $\sigma(X, X^*) \subset \|\cdot\|$-topology with equality if and only if $\dim X < \infty$.
3. $\sigma(X^*, X) \subset \sigma(X^*, X^{**}) \subset \|\cdot\|$-topology with equality in the second inclusion if and only if $\dim X < \infty$, and with equality in the first inclusion if and only if X is reflexive (*cf.* Proposition 5 below).
4. If Y is a subspace of X, then $\sigma(X, X^*) \restriction_Y = \sigma(Y, Y^*)$. This follows from Remark 4 on page 10 and from the Hahn–Banach theorem.

Similarly, we have $\sigma(X^{**}, X^*)\restriction_X = \sigma(X, X^*)$, *i.e.*, the subspace topology on X induced by the w^*-topology of X^{**} is the weak topology of X. Thus, the canonical embedding $X \to X^{**}$ is a w-to-w^*-homeomorphism from X onto its image (as well as being an isometric isomorphism).

Proposition 5. Let X be a normed space. Then

(i) a linear functional f on X is weakly continuous (*i.e.*, continuous with respect to the weak topology on X) if and only if $f \in X^*$. Briefly, $(X, w)^* = X^*$;

(ii) a linear functional φ on X^* is w^*-continuous (*i.e.*, continuous with respect to the weak-star topology on X^*) if and only if $\varphi \in X$, *i.e.*, there exists $x \in X$ with $\varphi = \hat{x}$. Briefly, $(X^*, w^*)^* = X$.

It follows that on X^* the weak and weak-star topologies coincide, *i.e.*, we have $\sigma(X^*, X^{**}) = \sigma(X^*, X)$ if and only if X is reflexive.

Definition. Let X be a normed space.
A set $A \subset X$ is *weakly bounded* if $\{f(x) : x \in A\}$ is bounded for all $f \in X^*$. Equivalently, for each w-open ngbd U of 0, there exists $\lambda > 0$ such that $A \subset \lambda U$.
A set $B \subset X^*$ is *weak-star bounded* if $\{f(x) : f \in B\}$ is bounded for all $x \in X$. Equivalently, for each w^*-open ngbd U of 0, there exists $\lambda > 0$ such that $B \subset \lambda U$.

Principal of Uniform Boundedness (PUB). If X is a Banach space, Y is a normed space, $\mathcal{T} \subset \mathcal{B}(X, Y)$ is pointwise bounded ($\sup\{\|Tx\| : T \in \mathcal{T}\} < \infty$ for all $x \in X$), then $\mathcal{T}$ is uniformly bounded ($\sup\{\|T\| : T \in \mathcal{T}\} < \infty$).

Proposition 6. Let X be a normed space.

(i) Any weakly bounded set in X is bounded in norm.

(ii) If X is complete then any weak-star bounded set in X^* is bounded in norm.

Notation. 1. We write $x_n \xrightarrow{w} x$ and say x_n *converges weakly to* x if $x_n \to x$ in the weak topology (in some normed space X). This happens if and only if $\langle x_n, f\rangle \to \langle x, f\rangle$ for all $f \in X^*$.
2. We write $f_n \xrightarrow{w^*} f$ and say f_n *converges weak-star to* f if $f_n \to f$ in the weak-star topology (in some dual space X^*). This happens if and only if $\langle x, f_n\rangle \to \langle x, f\rangle$ for all $x \in X$.

A consequence of PUB. Let X be a Banach space, Y a normed space, and (T_n) a sequence in $\mathcal{B}(X, Y)$ that converges pointwise to a function T. Then $T \in \mathcal{B}(X, Y)$, $\sup\|T_n\| < \infty$ and $\|T\| \leqslant \liminf\|T_n\|$.

Proposition 7. Let X be a normed space.

(i) If $x_n \xrightarrow{w} x$ in X, then $\sup\|x_n\| < \infty$ and $\|x\| \leqslant \liminf\|x_n\|$.

(ii) If $f_n \xrightarrow{w^*} f$ in X^* and X is complete, then $\sup\|f_n\| < \infty$ and $\|f\| \leqslant \liminf\|f_n\|$.

The Hahn–Banach Separation Theorems. Let $(X, \mathcal{P})$ be a locally convex space. Let C be a convex subset of X with $0 \in \operatorname{int} C$. Define

$$\mu_C \colon X \to \mathbb{R} \ , \qquad \mu_C(x) = \inf\{t > 0 : \, x \in tC\} \ .$$

This function is well-defined and is called the *Minkowski functional* or *gauge functional* of C.

Example. If X is a normed space and $C = B_X$, then $\mu_C = \|\cdot\|$.

Lemma 8. The function μ_C is a positive homogeneous, subadditive functional. Moreover, we have

$$\{x \in X : \, \mu_C(x) < 1\} \subset C \subset \{x \in X : \, \mu_C(x) \leqslant 1\}$$

with equality in the first inclusion when C is open and equality in the second inclusion when C is closed.

Remark. If C is symmetric in the case of real scalars, or balanced in the case of complex scalars, then μ_C is a seminorm. If in addition C is bounded, then μ_C is a norm.

Theorem 9. (Hahn–Banach separation theorem) Let $(X, \mathcal{P})$ be a locally convex space and C be an open convex subset of X with $0 \in C$. Let $x_0 \in X \setminus C$. Then there exists $f \in X^*$ such that $f(x_0) > f(x)$ for all $x \in C$ if the scalar field is $\mathbb{R}$, and $\operatorname{Re} f(x_0) > \operatorname{Re} f(x)$ for all $x \in C$ if the scalar field is $\mathbb{C}$.

Remark. From now on, for the rest of this chapter, we assume that the scalar field is $\mathbb{R}$. It is straightforward to modify the statements of theorems, their proofs, etc, in the case of complex scalars.

Theorem 10. (Hahn–Banach separation theorem) Let $(X, \mathcal{P})$ be a locally convex space and A, B be non-empty disjoint convex subsets of X.

(i) If A is open, then there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(x) < \alpha \leqslant f(y)$ for all $x \in A$, $y \in B$.

(ii) If A is compact and B is closed, then there exists $f \in X^*$ such that $\sup_A f < \inf_B f$.

Theorem 11. (Mazur) Let C be a convex subset of a normed space X. Then $\overline{C}^{\|\cdot\|} = \overline{C}^w$, *i.e.*, the norm-closure and weak-closure of C are the same. It follows that C is norm-closed if and only if C is weakly closed.

Corollary 12. Assume that $x_n \xrightarrow{w} 0$ in a normed space X. Then for all $\varepsilon > 0$ there exists $x \in \operatorname{conv}\{x_n : \, n \in \mathbb{N}\}$ with $\|x\| < \varepsilon$.

Remark. It follows that there exist $p_1 < q_1 < p_2 < q_2 < \dots$ and convex combinations $\sum_{i=p_n}^{q_n} t_i x_i$ that converge to zero in norm. In some but not all cases a stronger conclusion can be obtained: see Examples Sheet 2, Question 13.

Theorem 13. (Banach–Alaoglu) The dual ball B_{X^*} is w^*-compact for any normed space X.

Proposition 14. Let X be a normed space and K be a compact Hausdorff space.

(i) X is separable if and only if (B_{X^*}, w^*) is metrizable.

(ii) $C(K)$ is separable if and only if K is metrizable.

Remarks. 1. If X is separable, then (B_{X^*}, w^*) is a compact metric space. In particular, B_{X^*} is w^*-sequentially compact.
2. If X is separable, then X^* is w^*-separable. It is an easy consequence of Mazur's theorem that X is separable if and only if X is weakly separable. Thus, the previous statement reads: if X is w-separable, then X^* is w^*-separable. The converse is false, *e.g.,* for $X = \ell_\infty$ (see the remark following Goldstine's theorem).
3. The proof shows that any compact Hausdorff space K is a subspace of $(B_{C(K)^*}, w^*)$.
4. The proof shows that every normed space X embeds isometrically into $C(K)$ for some compact Hausdorff space. In particular, this holds with $K = (B_{X^*}, w^*)$ (*cf.* Theorem 19 below).

Proposition 15. X^* is separable if and only if (B_X, w) is metrizable.

Theorem 16. (Goldstine) $\overline{B_X}^{w^*} = B_{X^{**}}$, *i.e.*, the w^*-closure of the unit ball B_X of a normed space X in the second dual X^{**} is $B_{X^{**}}$.

Remark. It follows from Goldstine that $\overline{X}^{w^*} = X^{**}$. Thus, if X is separable, then X^{**} is w^*-separable. For example, ℓ_∞^* is w^*-separable.

Theorem 17. Let X be a Banach space. Then TFAE.

(i) X is reflexive.

(ii) (B_X, w) is compact.

(iii) X^* is reflexive.

Remark. It follows that if X is separable and reflexive, then (B_X, w) is a compact metric space, and hence sequentially compact.

Lemma 18. For every non-empty compact metric space K there is a continuous surjection $\varphi\colon \Delta \to K$, where $\Delta = \{0,1\}^{\mathbb{N}}$ with the product topology.

Note. Δ is homeomorphic via the map $(\varepsilon_i) \mapsto \sum_i (2\varepsilon_i)3^{-i}$ to the middle-third Cantor set.

Theorem 19. Every separable Banach space embeds isometrically into $C[0,1]$. Thus the separable space $C[0,1]$ is isometrically universal for the class of separable Banach spaces.

3 Schauder bases

Definition. A *Schauder basis* or simply *basis* of a Banach space X is a sequence (e_n) in X such that for all $x \in X$ there exists a unique sequence (a_n) of scalars such that x is the norm-convergent sum $x = \sum_{n=1}^{\infty} a_n e_n$.

Notation. For a sequence (e_n) in X, we let $[e_n] = \overline{\text{span}}\{e_n : n \in \mathbb{N}\}$.

Note. If (e_n) is a basis of X, then (e_n) is linearly independent and $X = [e_n]$. Thus, X is separable and infinite-dimensional.

Definition. A sequence (e_n) in a Banach space X is called a *basic sequence* if it is a basis of its closed linear span $[e_n]$.

Theorem 1. Let (e_n) be a sequence in a Banach space X. Then (e_n) is a basis of X if and only if the following three conditions hold:

(i) $e_n \neq 0$ for all $n \in \mathbb{N}$.

(ii) There exists $C \geqslant 1$ such that

$$\Big\| \sum_{i=1}^{m} a_i e_i \Big\| \leqslant C \Big\| \sum_{i=1}^{n} a_i e_i \Big\|$$

for all scalar sequences (a_i) and all $1 \leqslant m \leqslant n$ in $\mathbb{N}$.

(iii) $X = [e_n]$.

It follows that (e_n) is a basic sequence if and only if conditions (i) and (ii) hold.

Remark. If (e_n) is a basis of X, then the projections $P_n \colon X \to X$ defined by $P_n\big(\sum_{i=1}^{\infty} a_i e_i\big) = \sum_{i=1}^{n} a_i e_i$ are called the *basis projections.* By the above theorem, $\{P_n : n \in \mathbb{N}\}$ is uniformly bounded. The constant $C = \sup_{n \in \mathbb{N}} \|P_n\|$ is called the *basis constant* of (e_n), and it is the least C that satisfies condition (ii) above. A basis with basis constant 1 is called a *monotone basis.*

Examples. 1. The *Schauder system* is a monotone basis of $C[0, 1]$.

2. The *unit vector basis* is a monotone basis of ℓ_p, $1 \leqslant p < \infty$, and of c_0.

3. The *Haar system* is a monotone basis of $L_p[0, 1]$ for $1 \leqslant p < \infty$.

Remark. A famous result of Per Enflo shows that there are separable infinite-dimensional Banach spaces without bases. However, we have the following.

Theorem 2. (Mazur) Every infinite-dimensional Banach space contains a basic sequence. More precisely, for any $\varepsilon > 0$, every infinite-dimensional Banach space contains a normalised basic sequence with basis constant at most $1 + \varepsilon$.

Lemma 3. Let F be a finite-dimensional subspace of an infinite-dimensional Banach space X, and let $\varepsilon > 0$. Then there exists $y \in S_X$ such that $\|x\| \leqslant (1 + \varepsilon)\|x + \lambda y\|$ for all $x \in F$ and all scalars λ.

Remark. The proof of the lemma shows that there is a finite-codimensional subspace Y of X such that any $y \in S_Y$ works.

Dual bases. Let (e_n) be a basis of a Banach space X, let P_n, $n \in \mathbb{N}$, be the corresponding basis projections, and let C be the basis constant. For $n \in \mathbb{N}$, define the n^{th} coordinate functional e_n^* by

$$\Big\langle \sum a_i e_i, e_n^* \Big\rangle = a_n \ .$$

Then $e_n^* \in X^*$ with $\frac{1}{\|e_n\|} \leqslant \|e_n^*\| \leqslant \frac{2C}{\|e_n\|}$. Moreover,

$$P_n^*(x^*) = \sum_{i=1}^{n} x^*(e_i) e_i^* \qquad (x^* \in X^*,\ n \in \mathbb{N})$$

It follows that (e_i^*) is a basic sequence in X^* whose n^{th} basis projection is the restriction to $[e_i^*]$ of P_n^*, and hence its basis constant is at most C. We call (e_i^*) the *dual basis* or *biorthogonal sequence* of (e_i).

Equivalence of bases and perturbation. We begin with a simple result.

Proposition 4. For two basic sequences (e_n) and (f_n) in possibly different Banach spaces the following conditions are equivalent.

(i) $\sum a_i e_i$ converges if and only if $\sum a_i f_i$ converges for all (a_i).

(ii) The map $T(e_n) = f_n$ extends to an isomorphism $T\colon [e_n] \to [f_n]$.

(iii) There exist constants $A > 0$ and $B > 0$ such that

$$A\Big\| \sum a_i e_i \Big\| \leqslant \Big\| \sum a_i f_i \Big\| \leqslant B \Big\| \sum a_i e_i \Big\|$$

for all $(a_i) \in c_{00}$.

Definition. Two basic sequences (e_n) and (f_n) in possibly different Banach spaces are *equivalent*, written $(e_n) \sim (f_n)$, if any of the conditions (i), (ii) or (iii) in the proposition above is satisfied. More precisely, we say that (e_n) and (f_n) are *C-equivalent*, written $(e_n) \underset{C}{\sim} (f_n)$, if (iii) holds with $A/B \leqslant C$, or equivalently, if (ii) holds with $\|T\|\|T^{-1}\| \leqslant C$.

Proposition 5. Let (e_n) be a basic sequence in a Banach space X with dual basis (e_n^*). If a sequence (f_n) in X satisfies

$$\sum_{n=1}^{\infty} \|e_n^*\|\|e_n - f_n\| < 1 \tag{1}$$

then (f_n) is a basic sequence which is C-equivalent to (e_n), where $C = \frac{1+\gamma}{1-\gamma}$ and $\gamma = \sum_n \|e_n^*\|\|e_n - f_n\|$. Moreover, if $[e_n]$ is complemented in X, then so is $[f_n]$.

Block bases. Let (e_n) be a basis of a Banach space X with basis constant C. For $x = \sum_n a_n e_n \in X$, we define the *support* of X (with respect to (e_n)) to be the set

$$\operatorname{supp}(x) = \{n \in \mathbb{N} :\ a_n \neq 0\} \ .$$

Note that $\operatorname{supp}(x) = \{n \in \mathbb{N} :\ e_n^*(x) \neq 0\}$ where (e_n^*) is the dual basis.

For subsets A, B of $\mathbb{N}$, we write $A < B$ if $a < b$ for all $a \in A$ and $b \in B$. For vectors $x, y \in X$, we write $x < y$ if $\operatorname{supp}(x) < \operatorname{supp}(y)$. A *block basis* of (e_n) is a sequence $u_1 < u_2 < \dots$ of non-zero vectors in X. Note that (u_n) is then a basic sequence with basis constant at most C.

A sequence (x_n) in a Banach space X is *seminormalised* if there exist constants $a > 0$ and $b > 0$ such that $a \leqslant \|x_n\| \leqslant b$ for all $n \in \mathbb{N}$.

Proposition 6. Let (e_n) be a basis of a Banach space X with basis projections P_n, $n \in \mathbb{N}$. Let (x_n) be a seminormalised sequence in X such that $P_k(x_n) \to 0$ as $n \to \infty$ for each $k \in \mathbb{N}$. Then (x_n) has a basic subsequence equivalent to some block basis of (e_n).

Proposition 7. Let X be either ℓ_p for some $1 \leqslant p < \infty$ or c_0. Let (u_n) be a seminormalised block basis of the unit vector basis (e_n) of X. Then $(u_n) \sim (e_n)$ and $[u_n]$ is complemented in X.

Lemma 8. Let X be either ℓ_p for some $1 \leqslant p < \infty$ or c_0. Then every closed, infinite-dimensional subspace Y of X contains a further subspace Z that is isomorphic to X and complemented in X.

Definition. An infinite-dimensional Banach space is said to be *prime* if every infinite-dimensional complemented subspace of X is isomorphic to X.

Theorem 9. (Pełczyński) The Banach spaces ℓ_p, $1 \leqslant p < \infty$, and c_0 are prime.

Remark. The proof uses the *Pełczyński Decomposition Method.*

Theorem 10. (Pitt) Let $1 \leqslant p < q < \infty$. Then $\mathcal{B}(\ell_q, \ell_p) = \mathcal{K}(\ell_q, \ell_p)$, *i.e.*, every operator from ℓ_q to ℓ_p is compact. Similarly, $\mathcal{B}(c_0, \ell_p) = \mathcal{K}(c_0, \ell_p)$.

Definition. Let X be a normed space and Z be a subspace of X^*. Given $c > 0$, we say Z is *c-norming* for X if

$$c\|x\| \leqslant \sup\{|z^*(x)| : z^* \in B_Z\} \qquad \text{for all } x \in X.$$

We say Z is *norming* for X, if it is c-norming for X for some $c > 0$.

Note. X^* is 1-norming for X by Hahn–Banach.

Proposition 11. Let K be a seminormalised subset of a Banach space X, and let Z be a subspace of X^* that is norming for X. If $0 \in \overline{K}^{\sigma(X,Z)}$, then K contains a basic sequence.

Note. $\overline{K}^{\sigma(X,Z)}$ is the closure of K in the locally convex space $(X, \sigma(X, Z))$. Taking $Z = X^*$, if $0 \in \overline{K}^{w}$, then K contains a basic sequence.

Lemma 12. If $\varphi \in X^{**} \setminus X$, then $\ker \varphi$ is norming for the Banach space X.

Theorem 13. (Eberlein–Šmulian) Let X be a Banach space and $K \subset X$. Then the following are equivalent.

(i) $\overline{K}^{w}$ is weakly compact (*i.e.*, K is relatively weakly compact).

(ii) Every sequence in K has a subsequence that converges weakly in X.

In particular, K is weakly compact $\Leftrightarrow$ K is weakly sequentially compact.

4 Convexity

Let X be a real (or complex) vector space and K be a convex subset of X. We say a point $x \in K$ is an *extreme point of* K if whenever $x = (1-t)y + tz$ for some $y, z \in K$ and $t \in (0,1)$, then $y = z = x$. We denote by $\operatorname{Ext} K$ the set of all extreme points of K.

Examples. $\operatorname{Ext} B_{\ell_1^2} = \{\pm e_1, \pm e_2\}$, $\operatorname{Ext} B_{\ell_2^2} = S_{\ell_2^2}$, $\operatorname{Ext} B_{c_0} = \emptyset$.

Theorem 1. (Krein–Milman) Let $(X, \mathcal{P})$ be a locally convex space and K a compact convex subset of X. Then $K = \overline{\operatorname{conv}} \operatorname{Ext} K$. In particular, $\operatorname{Ext} K \neq \emptyset$ provided K is not empty.

Corollary 2. Let X be a normed space. Then $B_{X^*} = \overline{\operatorname{conv}}^{w^*} \operatorname{Ext} B_{X^*}$. In particular, $\operatorname{Ext} B_{X^*} \neq \emptyset$.

Remark. It follows that c_0 is not a dual space, *i.e.*, there is no normed space X with $c_0 \cong X^*$.

Definition. Let $(X, \mathcal{P})$ be a locally convex space and $K \subset X$ be a non-empty convex compact set. A *face of* K is a non-empty convex compact subset F of K such that for all $y, z \in K$ and $t \in (0,1)$, if $(1-t)y + tz \in F$, then $y, z \in F$.

Examples. 1. K is a face of K. For $x \in K$, we have $x \in \operatorname{Ext} K$ if and only if $\{x\}$ is a face of K.
2. For $f \in X^*$ and $\alpha = \sup_K f$, the set $F = \{x \in K : f(x) = \alpha\}$ is a face of K. Note that throughout this chapter we will use real scalars in statements of results and in their proofs. Obvious modifications yield the complex case, so here for example one would replace f by $\operatorname{Re} f$ in the definition of α and F.
3. If F is a face of K and E is a face of F, then E is a face of K. In particular, if F is a face of K and $x \in \operatorname{Ext} F$, then $x \in \operatorname{Ext} K$.

Definition. A linear map $T\colon X \to Y$ between Banach spaces is *weakly compact* if $\overline{TB_X}$ is weakly compact.

Note. By Mazur's theorem (Theorem 2.11) $\overline{TB_X} = \overline{TB_X}^w$. So, by the Eberlein–Šmulian theorem, T is weakly compact if and only if for every bounded sequence (x_n) in X, the sequence (Tx_n) has a subsequence that converges weakly in Y. Note also that a weakly compact linear map is bounded by Proposition 2.6.

Proposition 3. For a bounded linear map $T\colon X \to Y$ between Banach spaces the following are equivalent.

(i) T is weakly compact.

(ii) $T^{**}(X^{**}) \subset Y$.

(iii) $T^*\colon Y^* \to X^*$ is w^*-w continuous.

(iv) T^* is weakly compact.

Note. $\mathcal{W}(X, Y) = \{T \in \mathcal{B}(X, Y) : T \text{ is } w\text{-compact}\}$ is a closed subspace of $\mathcal{B}(X, Y)$ with the ideal property: $ATB \in \mathcal{W}(W, Z)$ whenever $A \in \mathcal{B}(Y, Z)$, $T \in \mathcal{W}(X, Y)$ and $B \in \mathcal{B}(W, X)$.

Theorem 4. (Krein–Šmulian) Let K be a w-compact subset of a Banach space. Then $\overline{\text{conv}}K$ is also w-compact.

Definition. A Banach space X is said to be *uniformly convex* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in B_X$ if $\|x-y\| \geqslant \varepsilon$, then $\left\|\frac{x+y}{2}\right\| \leqslant 1-\delta$.

Note. This definition is only meaningful for $\varepsilon \in (0, 2]$.

Examples. 1. Any Hilbert space is uniformly convex with $\delta = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$.

2. ℓ_1^2 is not uniformly convex. Similarly, ℓ_1, c_0, ℓ_∞, $L_1[0,1]$ and $L_\infty[0,1]$ are not uniformly convex.

3. Given $p_n \searrow 1$, the space $X = \left(\bigoplus_{n=1}^\infty \ell_{p_n}^2\right)_{\ell_2}$ is strictly convex, which means that $\left\|\frac{x+y}{2}\right\| < 1$ for $x \neq y$ and B_X, but not uniformly convex.

Proposition 5. Let $(\Omega, \mathcal{F}, \mu)$ be an arbitrary measure space and $2 \leqslant p < \infty$. Then $L_p(\mu)$ is uniformly convex.

Note. $L_p(\mu)$ is also uniformly convex in the range $1 < p < 2$.

Theorem 6. (Milman–Pettis) A uniformly convex Banach space is reflexive.

Remarks. 1. The converse is false. *E.g.,* if $p_n \searrow 1$, then $X = \left(\bigoplus_{n=1}^\infty \ell_{p_n}^2\right)_{\ell_2}$ is reflexive but not uniformly convex.

2. We are now in a position to complete the proof of Theorem1.12.

Theorem 7. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $1 \leqslant p < \infty$ and let $1 < q \leqslant \infty$ be the index conjugate to p.

(i) If $1 < p < \infty$, then L_p is reflexive and $L_p^* \cong L_q$.

(ii) If $p = 1$ and in addition μ is σ-finite, then $L_1^* \cong L_\infty$.

5 Banach algebras

A real or complex *algebra* is a real or, respectively, complex vector space A with a multiplication $A \times A \to A$, $(a, b) \mapsto ab$ satisfying

(i) $(ab)c = a(bc)$ for all $a, b, c \in A$;

(ii) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in A$;

(iii) $(\lambda a)b = a(\lambda b) = \lambda(ab)$ for all $a, b \in A$ and all scalars λ.

The algebra A is *unital* if there is an element $\mathbf{1} \in A$ such that $\mathbf{1} \neq 0$ and $a\mathbf{1} = \mathbf{1}a = a$ for all $a \in A$. This element is unique and is called the *unit of A*.

An *algebra norm on A* is a norm $\|\cdot\|$ on A such that $\|ab\| \leqslant \|a\|\|b\|$ for all $a, b \in A$. Note that multiplication is continuous with respect to the norm (as are addition and scalar multiplication). A *normed algebra* is an algebra together with an algebra norm on it. A *Banach algebra* is a complete normed algebra.

A *unital normed algebra* is a normed algebra A with an element $\mathbf{1} \in A$ such that $\|\mathbf{1}\| = 1$ and $a\mathbf{1} = \mathbf{1}a = a$ for all $a \in A$. Note that if A is a normed algebra containing an element $\mathbf{1} \neq 0$ such that $a\mathbf{1} = \mathbf{1}a = a$ for all $a \in A$, then

$$|||a||| = \sup\{\|ab\| : \|b\| \leqslant 1\}$$

defines an equivalent norm on A that makes A a unital normed algebra. A *unital Banach algebra* is a complete unital normed algebra.

A linear map $\theta \colon A \to B$ between algebras A and B is a *homomorphism* if $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in A$. If A and B are unital algebras with units $\mathbf{1}_A$ and $\mathbf{1}_B$, respectively, and in addition $\theta(\mathbf{1}_A) = \mathbf{1}_B$ then θ is called a *unital homomorphism.* In the category of normed algebras, an *isomorphism* is a continuous homomorphism with a continuous inverse, however, homomorphisms will not be assumed continuous.

Note. From now on the scalar field is always the field of complex numbers.

Examples. 1. If K is a compact Hausdorff space, then $C(K)$ is a commutative unital Banach algebra with pointwise multiplication and the uniform norm.
2. Let K be a compact Hausdorff space. A *uniform algebra on* K is a closed subalgebra of $C(K)$ that contains the constant functions and separates the points of K. For example, the *disc algebra*

$$A(\Delta) = \big\{f \in C(\Delta) : f \text{ is holomorphic on the interior of } \Delta\big\}$$

is a uniform algebra on the unit disc $\Delta = \{z \in \mathbb{C} : |z| \leqslant 1\}$. More generally, for a non-empty compact subset K of $\mathbb{C}$, the following are uniform algebras on K:

$$\mathcal{P}(K) \subset \mathcal{R}(K) \subset \mathcal{O}(K) \subset A(K) \subset C(K)$$

where $\mathcal{P}(K)$, $\mathcal{R}(K)$ and $\mathcal{O}(K)$ are the closures in $C(K)$ of, respectively, polynomials, rational functions without poles in K, and functions holomorphic on some open neighbourhood of K, whereas $A(K)$ is the algebra of continuous functions on K that are holomorphic on the interior of K. We will later see that $\mathcal{R}(K) = \mathcal{O}(K)$ always holds, whereas $\mathcal{R}(K) = \mathcal{P}(K)$ if and only if $\mathbb{C} \setminus K$ is connected (Runge's theorem). In general, $\mathcal{R}(K) \neq A(K)$, and $A(K) = C(K)$ if and only if K has empty interior.
3. $L_1(\mathbb{R})$ with the L_1-norm and convolution as multiplication defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,\mathrm{d}t$$

is a commutative Banach algebra without a unit (Riemann–Lebesgue lemma).
4. If X is a Banach space, then the algebra $\mathcal{B}(X)$ of bounded linear maps on X is a unital Banach algebra with composition as multiplication and the operator norm. It is non-commutative if $\dim X > 1$. An important special case is when X is a Hilbert space, in which case $\mathcal{B}(X)$ is a C^*-algebra (see Chapter 7).

Elementary constructions. 1. If A is a unital algebra with unit $\mathbf{1}$, then a *unital subalgebra* of A is a subalgebra B of A with $\mathbf{1} \in B$. If A is a normed algebra, then the closure of a subalgebra of A is a subalgebra of A.
2. The *unitization* of an algebra A is the vector space direct sum $A_+ = A \oplus \mathbb{C}$ with multiplication $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$. Then A_+ is a unital algebra

with unit $\mathbf{1} = (0, 1)$ and A is isomorphic to the ideal $\{(a, 0) : a \in A\}$. Identifying this ideal with A, we can write $A_+ = \{a + \lambda\mathbf{1} : a \in A, \ \lambda \in \mathbb{C}\}$.

When A is a normed algebra, its unitization A_+ becomes a unital normed algebra with norm $\|a + \lambda\mathbf{1}\| = \|a\| + |\lambda|$ for $a \in A$, $\lambda \in \mathbb{C}$. Note that A is then a closed ideal of A_+. When A is a Banach algebra, so is A_+.

3. The closure of an ideal of a normed algebra A is an ideal of A. If J is a closed ideal of A, then the quotient algebra A/J is a normed algebra with the quotient norm. If A is a unital normed algebra and J is a closed proper ideal of A, then A/J is a unital normed algebra in the quotient norm.

4. The (Banach space) completion $\widetilde{A}$ of a normed algebra A is a Banach algebra with multiplication defined as follows. Given $a, b \in \widetilde{A}$ choose sequences (a_n) and (b_n) in A converging to a and b, respectively, and set $ab = \lim_{n\to\infty} a_n b_n$. One of course needs to check that this is well-defined and has the required properties.

5. Let A be a unital Banach algebra. For $a \in A$, the map $L_a : A \to A$, $b \mapsto ab$, is a bounded linear operator. The map $a \mapsto L_a : A \to \mathcal{B}(A)$ is an isometric unital homomorphism. It follows that every Banach algebra is isometrically isomorphic to a closed subalgebra of $\mathcal{B}(X)$ for some Banach space X.

Elementary spectral theory.

Lemma 1. Let A be a unital Banach algebra, and let $a \in A$. If $\|\mathbf{1} - a\| < 1$, then a is invertible, and moreover $\|a^{-1}\| \leqslant \frac{1}{1-\|\mathbf{1}-a\|}$.

Notation. We write $G(A)$ for the group of invertible elements of a unital algebra A.

Corollary 2. Let A be a unital Banach algebra.

(i) $G(A)$ is an open subset of A.

(ii) $x \mapsto x^{-1}$ is a continuous map on $G(A)$.

(iii) If (x_n) is a sequence in $G(A)$ and $x_n \to x \notin G(A)$, then $\|x_n^{-1}\| \to \infty$.

(iv) If $x \in \partial G(A)$, then there is a sequence (z_n) in A such that $\|z_n\| = 1$ for all $n \in \mathbb{N}$ and $z_n x \to 0$ and $x z_n \to 0$ as $n \to \infty$. It follows that x has no left or right inverse in A, or in any unital Banach algebra that contains A as a (not necessarily unital) subalgebra.

Definition. Let A be an algebra and $x \in A$. The *spectrum* $\sigma_A(x)$ *of* x *in* A is defined as follows: if A is unital, then

$$\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin G(A)\} ,$$

and if A is non-unital then $\sigma_A(x) = \sigma_{A_+}(x)$.

Examples. 1. If $A = M_n(\mathbb{C})$ then $\sigma_A(x)$ is the set of eigenvalues of x.

2. If $A = C(K)$, K compact Hausdorff, then $\sigma_A(f) = f(K)$, the set of values taken by the function f.

3. If X is a Banach space and $A = \mathcal{B}(X)$, then for $T \in A$ the spectrum of T in A has the usual meaning: $\lambda \in \sigma_A(T)$ if and only if $\lambda I - T$ is not an isomorphism.

Theorem 3. Let A be a Banach algebra and $x \in A$. Then $\sigma_A(x)$ is a non-empty compact subset of $\{\lambda \in \mathbb{C} : |\lambda| \leqslant \|x\|\}$.

Corollary 4. (Gelfand–Mazur) A complex unital normed division algebra is isometrically isomorphic to $\mathbb{C}$.

Definition. Let A be a Banach algebra and $x \in A$. The *spectral radius* $r_A(x)$ *of* x *in* A is defined as

$$r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\} \ .$$

This is well-defined by Theorem 3 and satisfies $r_A(x) \leqslant \|x\|$.

Note. If x, y are commuting elements of a unital algebra A, then xy is invertible if and only if x and y are both invertible.

Lemma 5. (The Spectral Mapping Theorem for polynomials.) Let A be a unital Banach algebra and let $x \in A$. Then for a complex polynomial $p = \sum_{k=0}^{n} a_k z^k$, we have

$$\sigma_A(p(x)) = \{p(\lambda) : \lambda \in \sigma_A(x)\} \ ,$$

where $p(x) = \sum_{k=0}^{n} a_k x^k$ and $x^0 = \mathbf{1}$.

Theorem 6. (The Beurling–Gelfand Spectral Radius Formula.) Let A be a Banach algebra and $x \in A$. Then $r_A(x) = \lim \|x^n\|^{1/n} = \inf \|x^n\|^{1/n}$.

Theorem 7. Let A be a unital Banach algebra, let B be a closed unital subalgebra of A, and let $x \in B$. Then

$$\sigma_B(x) \supset \sigma_A(x) \qquad \text{and} \qquad \partial\sigma_B(x) \subset \partial\sigma_A(x) \ .$$

It follows that $\sigma_B(x)$ is the union of $\sigma_A(x)$ together with some of the bounded components of $\mathbb{C} \setminus \sigma_A(x)$.

Proposition 8. Let A be a unital Banach algebra and C be a maximal commutative subalgebra of A. Then C is a closed unital subalgebra of A and $\sigma_C(x) = \sigma_A(x)$ for all $x \in C$.

Commutative Banach algebras.

Definition. A non-zero homomorphism $\varphi \colon A \to \mathbb{C}$ on an algebra A is called a *character* on A. We denote by Φ_A the set of all characters on A. Note that if A is unital, then $\varphi(\mathbf{1}) = 1$ for all $\varphi \in \Phi_A$.

Lemma 9. Let A be a Banach algebra and $\varphi \in \Phi_A$. Then φ is continuous and $\|\varphi\| \leqslant 1$. Moreover, if A is unital then $\|\varphi\| = 1$.

Lemma 10. Let A be a unital Banach algebra and J be a proper ideal of A. Then the ideal $\overline{J}$ is also proper. It follows that every maximal ideal of A is closed.

Notation. For an algebra A we denote by $\mathcal{M}_A$ the set of all its maximal ideals.

Theorem 11. Let A be a commutative unital Banach algebra. Then the map $\varphi \mapsto \ker\varphi$ is a bijection $\Phi_A \to \mathcal{M}_A$.

Corollary 12. Let A be a commutative unital Banach algebra and $x \in A$.

(i) $x \in G(A)$ if and only if $\varphi(x) \neq 0$ for all $\varphi \in \Phi_A$.

(ii) $\sigma_A(x) = \{\varphi(x) : \varphi \in \Phi_A\}$.

(iii) $r_A(x) = \sup\{|\varphi(x)| : \varphi \in \Phi_A\}$.

Corollary 13. Let A be a Banach algebra and let x and y be commuting elements of A. Then $r_A(x+y) \leqslant r_A(x) + r_A(y)$ and $r_A(xy) \leqslant r_A(x)r_A(y)$.

Examples. 1. Let K be a compact Hausdorff space and $A = C(K)$. Then $\Phi_A = \{\delta_k : k \in K\}$. (Recall that $\delta_k(f) = f(k)$ for $k \in K$ and $f \in C(K)$.)
2. Let $K \subset \mathbb{C}$ be non-empty and compact. Then $\Phi_{\mathcal{R}(K)} = \{\delta_w : w \in K\}$.
3. Let $A = A(\Delta)$ be the disc algebra. Then $\Phi_A = \{\delta_w : w \in \Delta\}$.
4. The *Wiener algebra* $W = \{f \in C(\mathbb{T}) : \sum_{n\in\mathbb{Z}} |\hat{f}_n| < \infty\}$ is a commutative unital Banach algebra under pointwise operations and norm $\|f\|_1 = \sum_{n\in\mathbb{Z}} |\hat{f}_n|$. It is isomorphic to the Banach algebra $\ell_1(\mathbb{Z})$ with pointwise vector space operations, the ℓ_1-norm, and convolution as multiplication: $(a * b)_n = \sum_{j+k=n} a_j b_k$. The characters on W are again given by point evaluations: $\Phi_W = \{\delta_z : z \in \mathbb{T}\}$. Hence by Corollary 12(i) we obtain Wiener's theorem: if $f \in C(\mathbb{T})$ has absolutely summing Fourier series and $f(z) \neq 0$ for all $z \in \mathbb{T}$, then $1/f$ also has absolutely summing Fourier series.

Definition. Let A be a commutative unital Banach algebra. Then Φ_A is a w^*-closed subset of B_{A^*}, and hence it is w^*-compact and Hausdorff. The w^*-topology on Φ_A is called the *Gelfand topology* and Φ_A with its Gelfand topology is the *spectrum of* A or the *character space* or the *maximal ideal space* of A.

For $x \in A$ define the *Gelfand transform* $\hat{x}\colon \Phi_A \to \mathbb{C}$ *of* x by $\hat{x}(\varphi) = \varphi(x)$. Thus $\hat{x}$ is the restriction to Φ_A of the image of x in the second dual A^{**} under the canonical embedding. Thus $\hat{x} \in C(\Phi_A)$. The map $x \mapsto \hat{x}\colon A \to C(\Phi_A)$ is called the *Gelfand map.*

Theorem 14. (The Gelfand Representation Theorem) Let A be a commutative unital Banach algebra. Then the Gelfand map $A \to C(\Phi_A)$ is a continuous unital homomorphism. For $x \in A$ we have

(i) $\|\hat{x}\|_\infty = r_A(x) \leqslant \|x\|$

(ii) $\sigma_{C(\Phi_A)}(\hat{x}) = \sigma_A(x)$

(iii) $x \in G(A)$ if and only if $\hat{x} \in G\big(C(\Phi_A)\big)$.

Note. In general, the Gelfand map need not be injective or surjective. Its kernel is

$$\{x \in A : \sigma_A(x) = \{0\}\} = \{x \in A : \lim\|x^n\|^{1/n} = 0\} = \bigcap_{\varphi\in\Phi_A} \ker\varphi = \bigcap_{M\in\mathcal{M}_A} M\ .$$

An element $x \in A$ with $\lim\|x^n\|^{1/n} = 0$ is called *quasi-nilpotent.* The intersection $\bigcap_{M\in\mathcal{M}_A} M$ is called the *Jacobson radical of* A and is denoted by $J(A)$. We say A is *semisimple* if $J(A) = \{0\}$, *i.e.*, precisely when the Gelfand map is injective.

6 Holomorphic functional calculus

Recall that for a non-empty open subset U of $\mathbb{C}$ we denote by $\mathcal{O}(U)$ the locally convex space of complex-valued holomorphic functions on U with the topology of local uniform convergence. The topology is induced by the family of seminorms $f \mapsto \|f\|_K = \sup_{z\in K}|f(z)|$ defined for each non-empty compact subset K of U. Note that $\mathcal{O}(U)$ is also an algebra under pointwise multiplication, which is continuous with respect to the topology of $\mathcal{O}(U)$. This is an example of a *Fréchet algebra*: a complete metrizable locally convex space which is also an algebra with a continuous multiplication.

Notation. We define elements e and u of $\mathcal{O}(U)$ by $e(z) = 1$ and $u(z) = z$ for all $z \in U$. Note that $\mathcal{O}(U)$ is a unital algebra with unit e.

Theorem 1. (Holomorphic Functional Calculus) Let A be a commutative, unital Banach algebra, let $x \in A$ and U be an open subset of $\mathbb{C}$ with $\sigma_A(x) \subset U$. Then there exists a unique continuous unital homomorphism $\Theta_x \colon \mathcal{O}(U) \to A$ such that $\Theta_x(u) = x$. Moreover, $\varphi\big(\Theta_x(f)\big) = f(\varphi(x))$ for all $\varphi \in \Phi_A$ and $f \in \mathcal{O}(U)$. It follows that $\sigma_A\big(\Theta_x(f)\big) = \{f(\lambda) : \lambda \in \sigma_A(x)\}$ for all $f \in \mathcal{O}(U)$.

Remark. Think of Θ_x as "evaluation at x" and write $f(x)$ for $\Theta_x(f)$. Then the conclusions above can be briefly expressed as $\varphi(f(x)) = f(\varphi(x))$ and $\sigma_A(f(x)) = f(\sigma_A(x))$. The requirement that $e(x) = \mathbf{1}$ and $u(x) = x$ implies that for a complex polynomial $p = \sum_{k=0}^n a_k z^k = \sum_{k=0}^n a_k u^k$ we have $p(x) = \sum_{k=0}^n a_k x^k$ as defined in Lemma 5.5. Thus, Holomorphic Functional Calculus can be thought of as a far-reaching generalization of the Polynomial Spectral Mapping Theorem.

Theorem 2. (Runge's Approximation Theorem) Let K be a non-empty compact subset of $\mathbb{C}$. Then $\mathcal{O}(K) = \mathcal{R}(K)$, *i.e.*, if f is a holomorphic function on some open set containing K, then for all $\varepsilon > 0$ there is a rational function r without poles in K such that $\|f - r\|_K < \varepsilon$. More precisely, given any set Λ containing exactly one point from each bounded component of $\mathbb{C} \setminus K$, if f is a holomorphic function on some open set containing K, then for all $\varepsilon > 0$ there is a rational function r whose poles lie in Λ such that $\|f - r\|_K < \varepsilon$.

Note. If $\mathbb{C} \setminus K$ is connected, then $\Lambda = \emptyset$ which yields the polynomial approximation theorem $\mathcal{O}(K) = \mathcal{P}(K)$.

Vector-valued integration. Let $[a, b]$ be a closed bounded interval in $\mathbb{R}$, let X be a Banach space and $f \colon [a, b] \to X$ a continuous function. We define the integral $\int_a^b f(t)\,\mathrm{d}t$ as follows. Let $\mathcal{D}_n : a = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n}^{(n)} = b$, $n \in \mathbb{N}$, be a sequence of dissections of $[a, b]$ with

$$|\mathcal{D}_n| = \max_{1\leqslant j\leqslant k_n} |t_j^{(n)} - t_{j-1}^{(n)}| \to 0 \qquad \text{as} \qquad n \to \infty\ .$$

It follows from uniform continuity of f that the limit

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} f(t_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})$$

exists in X and is independent of the sequence of dissections chosen. The integral

$\int_a^b f(t)\,\mathrm{d}t$ is defined to be this limit. It follows easily from the definition that

$$\varphi\left(\int_a^b f(t)\,\mathrm{d}t\right) = \int_a^b \varphi(f(t))\,\mathrm{d}t \qquad \text{for all } \varphi \in X^* \ .$$

Applying the above with a norming functional φ at the element $\int_a^b f(t)\,\mathrm{d}t$ of X, we obtain

$$\Big\|\int_a^b f(t)\,\mathrm{d}t\Big\| \leqslant \int_a^b \|f(t)\|\,\mathrm{d}t \ .$$

Now let $\gamma\colon [a,b] \to \mathbb{C}$ be a path, and $f\colon [\gamma] \to X$ be a continuous function. We then define the *integral of f along γ* by

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \ .$$

If Γ is a chain $(\gamma_1, \dots, \gamma_n)$, then the integral along Γ of a continuous function $f\colon [\Gamma] \to \mathbb{C}$ is defined to be the sum of the integrals along the paths γ_j, *i.e.*, $\int_\Gamma f(z)\,\mathrm{d}z = \sum_{j=1}^n \int_{\gamma_j} f(z)\,\mathrm{d}z$.

Using the results about Banach space valued integrals over an interval, we immediately obtain the following results about Banach space valued integrals along a chain Γ:

$$\varphi\left(\int_\Gamma f(z)\,\mathrm{d}z\right) = \int_\Gamma \varphi(f(z))\,\mathrm{d}z \qquad \text{for all } \varphi \in X^*$$

and

$$\Big\|\int_\Gamma f(z)\,\mathrm{d}z\Big\| \leqslant \ell(\Gamma) \cdot \sup_{z\in[\Gamma]} \|f(z)\|$$

where $\ell(\Gamma)$ denotes the length of Γ. These properties together with the Hahn–Banach theorem allow us to deduce Banach space valued versions of classical scalar theorems.

Vector-valued Cauchy's theorem. Let U be a non-empty open subset of $\mathbb{C}$, and let Γ be a cycle in U such that $n(\Gamma, w) = 0$ for all $w \notin U$ (Γ does not wind round any point outside U). If X is a Banach space and $f\colon U \to X$ is holomorphic, then

$$\int_\Gamma f(z)\,\mathrm{d}z = 0 \ .$$

This follows by applying a norming functional to the integral and by applying the scalar Cauchy's theorem.

Lemma 3. Let A, x, U be as in Theorem 1. Set $K = \sigma_A(x)$ and fix a cycle Γ in $U \setminus K$ such that

$$n(\Gamma, w) = \begin{cases} 1 & \text{if } w \in K \\ 0 & \text{if } w \notin U \end{cases}$$

Define $\Theta_x\colon \mathcal{O}(U) \to A$ by letting

$$\Theta_x(f) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma f(z)(z\mathbf{1} - x)^{-1}\,\mathrm{d}z$$

for $f \in \mathcal{O}(U)$. Then

(i) Θ_x is well-defined, linear and continuous.

(ii) For a rational function r without poles in U we have $\Theta_x(r) = r(x)$ in the usual sense.

(iii) $\varphi\big(\Theta_x(f)\big) = f(\varphi(x))$ for all $\varphi \in \Phi_A$ and $f \in \mathcal{O}(U)$, and hence

$$\sigma_A\big(\Theta_x(f)\big) = \{f(\lambda) : \lambda \in \sigma_A(x)\} \ .$$

Note. Lemma 3 can be viewed as a Banach algebra valued version of Cauchy's Integral Formula. It comes close to proving Theorem 1. What is missing is that Θ_x is multiplicative (which in fact follows from Lemma 3 for semisimple algebras) and that Θ_x is unique. Our strategy is to deduce Runge's theorem (Theorem 2) from Lemma 3 and then use it to complete the proof of Theorem 1 by first showing the following corollary of Runge's theorem.

Corollary 4. Let U be a non-empty open subset of $\mathbb{C}$, and let $\mathcal{R}(U)$ be the set of rational functions without poles in U. Then $\mathcal{R}(U)$ is dense in $\mathcal{O}(U)$ in the topology of local uniform convergence.

7 C^*-algebras

A $*$-*algebra* is a complex algebra A with an *involution*: a map $x \mapsto x^* \colon A \to A$ such that

(i) $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$ (ii) $(xy)^* = y^* x^*$ (iii) $x^{**} = x$

for all $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$. Note that if A is unital with unit $\mathbf{1}$, then $\mathbf{1}^* = \mathbf{1}$.

A C^*-*algebra* is a Banach algebra A with an involution that satisfies the C^*-*equation*:

$$\|x^* x\| = \|x\|^2 \qquad \text{for all } x \in A \ .$$

A complete algebra norm on a $*$-algebra that satisfies the C^*-equation is called a C^*-*norm*. Thus, a C^*-algebra is a $*$-algebra with a C^*-norm.

Remarks. 1. If A is a C^*-algebra, then $\|x^*\| = \|x\|$ for all $x \in A$. It follows that the involution is continuous. A *Banach* $*$-*algebra* is a Banach algebra A with an involution such that $\|x^*\| = \|x\|$ for all $x \in A$. Thus, every C^*-algebra is a Banach $*$-algebra.
2. A C^*-algebra which has a multiplicative identity $\mathbf{1} \neq 0$, is automatically a unital C^*-algebra, *i.e.*, $\|\mathbf{1}\| = 1$.

Definitions. A subalgebra B of a $*$-algebra A is a $*$-*subalgebra of* A if $x^* \in B$ for every $x \in B$. A closed $*$-subalgebra (called a C^*-*subalgebra*) of a C^*-algebra is a C^*-algebra. The closure of a $*$-subalgebra of a C^*-algebra A is a $*$-subalgebra of A, and hence a C^*-subalgebra of A. A homomorphism $\theta \colon A \to B$ between $*$-algebras is a $*$-*homomorphism* if $\theta(x^*) = \theta(x)^*$ for all $x \in A$. A $*$-*isomorphism* is a bijective $*$-homomorphism.

Examples. 1. Let K be a compact Hausdorff space. Then $C(K)$ is a commutative unital C^*-algebra with the uniform norm and with involution $f \mapsto f^*$, where $f^*(k) = \overline{f(k)}$ for $k \in K$, $f \in C(K)$.

2. Let H be a Hilbert space. Then the algebra $\mathcal{B}(H)$ of bounded linear operators on H is a C^*-algebra in the operator norm and involution $T \mapsto T^*$, where T^* is the adjoint of T defined by the equation $\langle Tx, y\rangle = \langle x, T^*y\rangle$, $x, y \in H$.
3. Every C^*-subalgebra of $\mathcal{B}(H)$ is a C^*-algebra.

Remark. The Gelfand–Naimark theorem states that for every C^*-algebra A there is a Hilbert space H and an isometric $*$-isomorphism between A and a C^*-subalgebra of $\mathcal{B}(H)$. We will prove only a commutative version of this (Theorem 4 below).

Definition. Let A be a C^*-algebra and $x \in A$. We say x is

(i) *hermitian* or *self-adjoint* if $x^* = x$;

(ii) *unitary* if (A is unital and) $x^*x = xx^* = \mathbf{1}$.

(iii) *normal* if $x^*x = xx^*$.

Examples. 1. The unit $\mathbf{1}$ is both hermitian and unitary. In general, hermitian and unitary elements are normal.
2. In $C(K)$ a function f is hermitian if and only if $f(K) \subset \mathbb{R}$ (*i.e.*, f has real spectrum), and f is unitary if and only if $f(K) \subset \mathbb{T}$. This will be generalized in Corollary 3 below.

Remarks. 1. For all $x \in A$ there exits unique hermitian elements h, k such that $x = h + \mathrm{i}k$. It follows that $x^* = h - \mathrm{i}k$, and that x is normal if and only if $hk = kh$.
2. If A is unital then $x \in G(A)$ if and only if $x^* \in G(A)$, in which case $(x^*)^{-1} = (x^{-1})^*$. It follows that $\sigma_A(x^*) = \{\bar{\lambda} : \lambda \in \sigma_A(x)\}$, and thus $r_A(x^*) = r_A(x)$.

Lemma 1. Let A be a C^*-algebra and $x \in A$ be normal. Then $r_A(x) = \|x\|$.

Lemma 2. Let A be a unital C^*-algebra. Then $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in A$ and for all $\varphi \in \Phi_A$.

Remark. The condition that A is unital is superfluous. However, unitization for a C^*-algebra is not quite straightforward (see Examples Sheet 4) and the above will suffice for us.

Corollary 3. Let A be a unital C^*-algebra.

(i) If $x \in A$ is hermitian then $\sigma_A(x) \subset \mathbb{R}$.

(ii) If $x \in A$ is unitary then $\sigma_A(x) \subset \mathbb{T}$.

If B is a unital C^*-subalgebra of A and $x \in B$ is normal, then $\sigma_A(x) = \sigma_B(x)$.

Theorem 4. (Commutative Gelfand–Naimark) Let A be a commutative unital C^*-algebra. Then A is isometrically $*$-isomorphic to $C(K)$ for some compact Hausdorff space K. More precisely, the Gelfand map $x \mapsto \hat{x} \colon A \to C(\Phi_A)$ is an isometric $*$-isomorphism.

Application: positive square roots. Let A be a unital C^*-algebra. We say that an element $x \in A$ is *positive* if x is hermitian and $\sigma_A(x) \subset [0, \infty)$.

Every positive element x of A has a unique positive square root: there exists a unique positive element $y \in A$ such that $y^2 = x$. The unique positive square root of x is denoted $x^{1/2}$.

Remark. This applies in particular to positive elements of $\mathcal{B}(H)$, H a Hilbert space. Recall that $T \in \mathcal{B}(H)$ is positive if and only if $\langle Tx, x \rangle \geqslant 0$ for all $x \in H$.

Application: polar decomposition. Let T be an invertible operator on a Hilbert space H. Then there exist unique operators R and U such that R is positive, U is unitary and $T = RU$.

8 Borel functional calculus and spectral theory

Throughout, H is a (non-zero) complex Hilbert space, $\mathcal{B}(H)$ is the C*-algebra of all bounded, linear operators on H, K is a compact Hausdorff space and $\mathcal{B}$ is the Borel σ-field on K.

Bounded Borel functions. Let $L_\infty(K)$ denote the set of all bounded Borel functions $f \colon K \to \mathbb{C}$. This is a commutative, unital C*-algebra equipped with the 'sup norm' $\|f\|_K$. The simple functions form a dense $*$-subalgebra of $L_\infty(K)$.

Theorem 1. (Spectral theorem for commutative C*-algebras.) Let A be a commutative, unital C*-subalgebra of $\mathcal{B}(H)$ and let $K = \Phi_A$. Then there is a unique (norm-decreasing) $*$-homomorphism $\Psi \colon L_\infty(K) \to \mathcal{B}(H)$ such that

(i) $\Psi(\widehat{T}) = T$ for all $T \in A$, where $\widehat{T}$ is the Gelfand transform of T, and

(ii) letting $P(E) = \Psi(\mathbf{1}_E)$ for $E \in \mathcal{B}$, the map

$$P_{x,y} \colon \mathcal{B} \to \mathbb{C} \ , \qquad P_{x,y}(E) = \langle P(E)x, y \rangle$$

is a regular complex Borel measure on K for every $x, y \in H$.

Moreover,

(iii) $P(U) \neq 0$ for every non-empty open subset U of K; and for $S \in \mathcal{B}(H)$,

(iv) S commutes with every $T \in A$ if and only if it commutes with every $P(E)$ $(E \in \mathcal{B})$.

Remark. We can think of the map $P \colon \mathcal{B} \to \mathcal{B}(H)$ as an operator-valued Borel 'measure' on K. For $f \in L_\infty(K)$, we define its integral $\int_K f \, \mathrm{d}P$ to be $\Psi(f)$. Note that for $E \in \mathcal{B}$, we have

$$\int_K \mathbf{1}_E \, \mathrm{d}P = \Psi(\mathbf{1}_E) = P(E)$$

is an orthogonal projection. Also, for $T \in A$, we have

$$\int_K \widehat{T} \, \mathrm{d}P = \Psi(\widehat{T}) = T \ .$$

Thus, we can think of the spectral theorem above as an operator-valued Riesz Representation Theorem.

Exponentials in Banach algebras. Let A be a unital Banach algebra. Then for $x \in A$ we define $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The sum converges absolutely, and hence it converges in A. It is easy to check that if $x, y \in A$ commute then $e^{x+y} = e^x e^y$.

Lemma 2. (Fuglede-Putnam-Rosenblum) Let A be a unital C*-algebra and $x, y, z \in A$. Assume that x and y are normal and $xz = zy$. Then $x^*z = zy^*$. In particular, if z commutes with x, then it commutes with x^*.

Theorem 3. (Spectral theorem for normal operators.) Let $T \in \mathcal{B}(H)$ be a normal operator. Then there is a unique operator-valued Borel measure P on $\sigma(T)$ such that

$$T = \int_{\sigma(T)} \lambda \, \mathrm{d}P \ .$$

Moreover, $S \in \mathcal{B}(H)$ commutes with every projection $P(E)$ $(E \in \mathcal{B})$ if and only if $ST = TS$.

Remarks. 1. Let $K = \sigma(T)$ and $z \in L_\infty(K)$ be the function $z(\lambda) = \lambda$ $(\lambda \in K)$. The first statement of the theorem above is shorthand for the following: there is a unique (norm-decreasing) unital $*$-homomorphism $\Psi \colon L_\infty(K) \to \mathcal{B}(H)$ such that $\Psi(z) = T$ and $P_{x,y}$ as defined in Theorem 1 is a regular complex Borel measure on K for all $x, y \in H$.

2. The integral representation of T is called the *spectral decomposition* of T. The orthogonal projections $P(E)$ $(E \in \mathcal{B})$ are the *spectral projections* of T.

Theorem 4. (Borel functional calculus for a normal operator) Let $T \in \mathcal{B}(H)$ be a normal operator and let $K = \sigma(T)$. For $f \in L_\infty(K)$ define

$$f(T) = \int_K f \, \mathrm{d}P \ ,$$

where P is the unique operator-valued Borel measure on $K = \sigma(T)$ given by Theorem 3. The map $f \mapsto f(T)$ has the following properties:

(i) it is a unital $*$-homomorphism from $L_\infty(K)$ to $\mathcal{B}(H)$ with $z(T) = T$ (where $z(\lambda) = \lambda, \ \lambda \in K$);

(ii) $\|f(T)\| \leqslant \|f\|_K$ for all $f \in L_\infty(K)$ with equality for $f \in C(K)$;

(iii) if $S \in \mathcal{B}(H)$ and $ST = TS$, then $Sf(T) = f(T)S$ for all $f \in L_\infty(K)$.

(iv) $\sigma(f(T)) \subset \overline{f(K)}$ for all $f \in L_\infty(K)$.

Polar decomposition of normal operators. Let $T \in \mathcal{B}(H)$ be a normal operator. Then $T = RU$, where R is positive, U is unitary, and R, U, T pairwise commute.

Representation of unitary operators. Let $U \in \mathcal{B}(H)$ be a unitary operator. Then $U = e^{iQ}$ for some hermitian operator Q.

Connectedness of $G(\mathcal{B}(H))$. The group of all invertible operators in $\mathcal{B}(H)$ is connected. Moreover, every invertible operator is the product of two exponentials.