Part III Functional Analysis

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1 The Hahn–Banach extension theorems

Dual space. Let X be a normed space. The dual space X^* of X is the space of all bounded linear functionals on X. The dual space is a Banach space in the operator norm which is defined for $f \in X^*$ as

$$||f|| = \sup \{|f(x)| : x \in B_X\}$$
.

Recall that $B_X = \{x \in X : ||x|| \le 1\}$ is the closed unit ball of X and that $S_X = \{x \in X : ||x|| = 1\}$ is the unit sphere of X.

Examples. $\ell_p^* \cong \ell_q$ for $1 \leqslant p < \infty$, $1 < q \leqslant \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We also have $c_0^* \cong \ell_1$. If H is a Hilbert space then $H^* \cong H$ (conjugate-linear in the complex case).

Notation. 1. Given normed spaces X and Y, we write $X \sim Y$ if X and Y are isomorphic, *i.e.*, when there is a linear bijection $T: X \to Y$ such that both T and T^{-1} are continuous. (Recall that by the Open Mapping Theorem, if X and Y are both complete and T is a continuous linear bijection, then T^{-1} is automatically continuous.)

- **2.** Given normed spaces X and Y, we write $X \cong Y$ if X and Y are isometrically isomorphic, *i.e.*, when there is a surjective linear map $T: X \to Y$ such that ||Tx|| = ||x|| for all $x \in X$. It follows that T is a continuous linear bijection and that T^{-1} is also isometric, and hence continuous.
- **3.** For $x \in X$ and $f \in X^*$, we shall sometimes denote f(x), the action of f on x, by $\langle x, f \rangle$. By definition of the operator norm we have

$$|\langle x, f \rangle| = |f(x)| \leqslant ||f|| \cdot ||x||.$$

When X is a Hilbert space and we identify X^* with X in the usual way, then $\langle x, f \rangle$ is the inner product of x and f.

Definition. Let X be a real vector space. A functional $p: X \to \mathbb{R}$ is called

- positive homogeneous if p(tx) = tp(x) for all $x \in X$ and $t \ge 0$, and
- subadditive if $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

Theorem 1. (Hahn–Banach) Let X be a real vector space and p be a positive homogeneous, subadditive functional on X. Let Y be a subspace of X and $g\colon Y\to\mathbb{R}$ be a linear functional such that $g(y)\leqslant p(y)$ for all $y\in Y$. Then there is a linear functional $f\colon X\to\mathbb{R}$ such that $f\!\upharpoonright_Y=g$ and $f(x)\leqslant p(x)$ for all $x\in X$.

Zorn's Lemma. This is used in the proof, so we recall it here. Let (P, \leq) be a poset (partially ordered set). An element $x \in P$ is an *upper bound* for a subset A of P if $a \leq x$ for all $a \in A$. A subset C of P is called a *chain* if it is linearly ordered by \leq . An element $x \in P$ is a *maximal element* of P if whenever $x \leq y$ for some $y \in P$, we have y = x. Zorn's Lemma states that if $P \neq \emptyset$ and every non-empty chain in P has an upper bound, then P has a maximal element.

Definition. A *seminorm* on a real or complex vector space X is a functional $p: X \to \mathbb{R}$ such that

- $p(x) \ge 0$ for all $x \in X$;
- $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and for all scalars λ ;
- $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

Note. "norm" \Longrightarrow "seminorm" \Longrightarrow "positive homogeneous+subadditive".

Theorem 2. (Hahn–Banach) Let X be a real or complex vector space and p be a seminorm on X. Let Y be a subspace of X and g be a linear functional on Y such that $|g(y)| \leq p(y)$ for all $y \in Y$. Then there is a linear functional f on X such that $f|_{Y} = g$ and $|f(x)| \leq p(x)$ for all $x \in X$.

Remark. It follows from the proof given in lectures that for a complex normed space X, the map $f \mapsto \operatorname{Re}(f) \colon \left(X^*\right)_{\mathbb{R}} \to \left(X_{\mathbb{R}}\right)^*$ is an isometric isomorphism. Here $Y_{\mathbb{R}}$, for a complex vector space Y, denotes the real vector space obtained from Y by restricting scalar multiplication to the reals.

Corollary 3. Let X be a real or complex vector space and p be a seminorm on X. For every $x_0 \in X$ there is a linear functional f on X with $f(x_0) = p(x_0)$ and $|f(x)| \leq p(x)$ for all $x \in X$.

Theorem 4. (Hahn–Banach) Let X be a real or complex normed space. Then

- (i) Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ with $f \upharpoonright_Y = g$ and ||f|| = ||g||.
- (ii) Given a non-zero $x_0 \in X$, there exists $f \in S_{X^*}$ with $f(x_0) = ||x_0||$.

Remarks. 1. Part (i) can be viewed as a linear version of Tietze's extension theorem. The latter states that if L is a closed subset of a compact Hausdorff space K, and $g: L \to \mathbb{R}$ (or \mathbb{C}) is continuous, then there is a continuous function $f: K \to \mathbb{R}$ (respectively, \mathbb{C}) such that $f|_{L} = g$ and $||f||_{\infty} = ||g||_{\infty}$.

- **2.** Part (ii) implies that X^* separates the points of X: given $x \neq y$ in X, there exists $f \in X^*$ such that $f(x) \neq f(y)$. Thus there are "plenty" bounded linear functionals on X.
- **3.** The element $f \in X^*$ in part (ii) is called a norming functional for x_0 . It shows that

$$||x_0|| = \max\{|\langle x_0, g \rangle| : g \in B_{X^*}\}.$$

Another name for f is support functional at x_0 . Assume X is a real normed space and that $||x_0|| = 1$. Then $B_X \subset \{x \in X : f(x) \leq 1\}$, and hence the hyperplane $\{x \in X : f(x) = 1\}$ can be thought of as a tangent to B_X at x_0 .

Bidual. Let X be a normed space. Then $X^{**} = (X^*)^*$ is called the *bidual* or second dual of X. It is the Banach space of all bounded linear functionals on X^* with the operator norm. For $x \in X$ we define $\hat{x} \colon X^* \to \mathbb{R}$ (or \mathbb{C}) by $\hat{x}(f) = f(x)$ (evaluation at x). Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leq \|x\|$. The map $x \mapsto \hat{x} \colon X \to X^{**}$ is called the canonical embedding.

Theorem 5. The canonical embedding defined above is an isometric isomorphism of X into X^{**} .

Remarks. 1. Using the bracket notation, we have

$$\langle f, \hat{x} \rangle = \langle x, f \rangle = f(x)$$
 $x \in X, f \in X^*$.

- **2.** The image $\widehat{X} = \{\widehat{x} : x \in X\}$ of the canonical embedding is closed in X^{**} if and only if X is complete.
- **3.** In general, the closure of \widehat{X} in X^{**} is a Banach space which contains an isometric copy of X as a dense subspace. We have thus proved that every normed space has a completion.

Definition. X is reflexive if the canonical embedding of X into X^{**} is surjective

Examples. 1. The spaces ℓ_p , $1 , Hilbert spaces, finite-dimensional spaces are reflexive. Later we prove that the spaces <math>L_p(\mu)$, 1 , are also reflexive.

2. The spaces c_0 , ℓ_1 , $L_1[0,1]$ are not reflexive.

Remark. Every reflexive space X is complete with $X \cong X^{**}$. There are Banach spaces X with $X \cong X^{**}$ which are not reflexive. So for $1 , it is not sufficient to say that <math>\ell_p^{**} \cong \ell_q^* \cong \ell_p$ (where q is the conjugate index of p) to deduce that ℓ_p is reflexive. One also has to verify that this isometric isomorphism is the canonical embedding.

Dual operators. Recall that for normed spaces X, Y we denote by $\mathcal{B}(X, Y)$ the space of all bounded linear maps $T \colon X \to Y$. This is a normed space in the operator norm:

$$||T|| = \sup \{||Tx|| : x \in B_X\}$$
.

Moreover, $\mathcal{B}(X,Y)$ is complete if Y is.

For $T \in \mathcal{B}(X,Y)$, the dual operator of T is the map $T^* \colon Y^* \to X^*$ defined by $T^*(g) = g \circ T$ for $g \in Y^*$. In bracket notation:

$$\langle x, T^*g \rangle = \langle Tx, g \rangle \qquad x \in X, \ g \in Y^* \ .$$

 T^* is linear and bounded with $||T^*|| = ||T||$ (uses Theorem 4(ii)).

Remark. When X and Y are Hilbert spaces, the dual operator T^* corresponds to the adjoint of T after identifying X^* and Y^* with X and Y, respectively.

Example. Let $1 and consider the right shift <math>R: \ell_p \to \ell_p$. Then $R^*: \ell_q \to \ell_q$ (q the conjugate index of p) is the left shift.

Properties of dual operators. 1. $(\operatorname{Id}_X)^* = \operatorname{Id}_{X^*}$

- **2.** $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ for scalars λ, μ and for $S, T \in \mathcal{B}(X, Y)$. Note that unlike for adjoints there is no complex conjugation here. That is because the identification of a Hilbert space with its dual is conjugate linear in the complex case.
- **3.** $(ST)^* = T^*S^*$ for $T \in \mathcal{B}(X,Y)$ and $S \in \mathcal{B}(Y,Z)$.
- **4.** The map $T \mapsto T^* : \mathcal{B}(X,Y) \to \mathcal{B}(Y^*,X^*)$ is an *into* isometric isomorphism.
- **5.** The following diagram

$$X \xrightarrow{T} Y$$

$$\downarrow \qquad \qquad \downarrow$$

$$X^{**} \xrightarrow{T^{**}} Y^{**}$$

commutes, where the vertical arrows represent canonical embeddings.

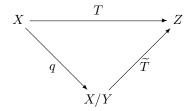
Remark. It follows from the above properties that if $X \sim Y$ then $X^* \sim Y^*$.

Quotient space. Let X be a normed space and Y a closed subspace. Then the quotient space X/Y becomes a normed space in the *quotient norm* defined as follows.

$$||x + Y|| = \inf \{||x + y|| : y \in Y\}$$
 $x \in X$.

The quotient map $q: X \to X/Y$ is a surjective linear map that is bounded with $||q|| \le 1$. It maps the open unit ball $D_X = \{x \in X : ||x|| < 1\}$ onto $D_{X/Y}$. It follows that ||q|| = 1 (if $X \ne Y$) and that q is an open map (it maps open sets onto open sets).

If $T: X \to Z$ is a bounded linear map with $Y \subset \ker(T)$, then there is a unique map $\widetilde{T}: X/Y \to Z$ such that the diagram



commutes, and moreover \widetilde{T} is linear, bounded and $\|\widetilde{T}\| = \|T\|$.

Theorem 6. Let X be a normed space. If X^* is separable, then so is X.

Remark. The converse is false. E.g., $X = \ell_1$ is separable but $X^* \cong \ell_{\infty}$ is not.

Theorem 7. Every separable normed space is isometrically isomorphic to a subspace of ℓ_{∞} .

Remarks. 1. The result says that ℓ_{∞} is isometrically universal for the class \mathcal{SB} of all separable Banach spaces. We will later see that there is a separable space with the same property.

2. A dual version of the above result states that every separable Banach spaces is a quotient of ℓ_1 .

Theorem 8. (Vector-valued Liouville) Let X be a complex Banach space and $f: \mathbb{C} \to X$ be a bounded, holomorphic function. Then f is constant.

Locally Convex Spaces. A locally convex space (LCS) is a pair (X, \mathcal{P}) , where X is a real or complex vector space and \mathcal{P} is a family of seminorms on X that separates the points of X in the sense that for every $x \in X$ with $x \neq 0$, there is a seminorm $p \in \mathcal{P}$ with $p(x) \neq 0$. The family \mathcal{P} defines a topology on X: a set $U \subset X$ is open if and only if for all $x \in U$ there exist $n \in \mathbb{N}, p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon > 0$ such that

$$\{y \in X : p_k(y-x) < \varepsilon \text{ for } 1 \leqslant k \leqslant n\} \subset U$$
.

Remarks. 1. Addition and scalar multiplication are continuous.

- **2.** The topology of X is Hausdorff as \mathcal{P} separates the points of X.
- **3.** A sequence $x_n \to x$ in X if and only if $p(x_n x) \to 0$ for all $p \in \mathcal{P}$. (The same holds for nets.)
- **4.** For a subspace Y of X define $\mathcal{P}_Y = \{p | Y : p \in \mathcal{P}\}$. Then (Y, \mathcal{P}_Y) is a locally convex space, and the corresponding locally convex topology on Y is nothing else but the subspace topology on Y induced by X.
- **5.** Let \mathcal{P} and \mathcal{Q} be two families of seminorms on X both of which separate the points of X. We say \mathcal{P} and \mathcal{Q} are *equivalent* and write $\mathcal{P} \sim \mathcal{Q}$ if they define the same topology on X.

The topology of a locally convex space (X, \mathcal{P}) is metrizable if and only if there is a countable \mathcal{Q} with $\mathcal{Q} \sim \mathcal{P}$.

Definition. A Fréchet space is a complete metrizable locally convex space.

Examples. 1. Every normed space $(X, \|\cdot\|)$ is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.

2. Let U be a non-empty, open subset of \mathbb{C} , and let $\mathcal{O}(U)$ denote the space of all holomorphic functions $f \colon U \to \mathbb{C}$. For a compact set $K \subset U$ and for $f \in \mathcal{O}(U)$ set $p_K(f) = \sup_{z \in K} |f(z)|$. Set $\mathcal{P} = \{p_K : K \subset U, K \text{ compact}\}$. Then $(\mathcal{O}(U), \mathcal{P})$ is a locally convex space whose topology is the topology of local uniform convergence.

There exist compact sets $K_n \subset U$, $n \in \mathbb{N}$, such that $K_n \subset \operatorname{int}(K_{n+1})$ and $U = \bigcup_n K_n$. Then $\{p_{K_n} : n \in \mathbb{N}\}$ is countable and equivalent to \mathcal{P} . Hence $(\mathcal{O}(U), \mathcal{P})$ is metrizable and in fact it is a Fréchet space.

The topology of local uniform convergence is not *normable*: the topology is not induced by a norm. This follows, for example, from Montel's theorem: given a sequence (f_n) in $\mathcal{O}(U)$ such that $(f_n \upharpoonright_K)$ is bounded in $(C(K), \|\cdot\|_{\infty})$ for every compact $K \subset U$, there is a subsequence of (f_n) that converges locally uniformly.

3. Fix $d \in \mathbb{N}$ and a non-empty open set $\Omega \subset \mathbb{R}^d$. Let $C^{\infty}(\Omega)$ denote the space of all infinitely differentiable functions $f \colon \Omega \to \mathbb{R}$. Every multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{Z}_{\geq 0})^d$ gives rise to a partial differential operator

$$D^{\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} .$$

For a multi-index α , a compact set $K \subset \Omega$, and $f \in \mathcal{O}(U)$ define

$$p_{K,\alpha}(f) = \sup \left\{ |(D^{\alpha}f)(x)| : x \in K \right\}.$$

Set $\mathcal{P} = \{p_{K,\alpha} : K \subset \Omega \text{ compact}, \alpha \in (\mathbb{Z}_{\geqslant 0})^d\}$. Then $(C^{\infty}(\Omega), \mathcal{P})$ is a locally convex space. It is a Fréchet space and is not normable.

Lemma 9. Let (X, \mathcal{P}) and (Y, \mathcal{Q}) be locally convex spaces. Let $T: X \to Y$ be a linear map. Then TFAE:

- (i) T is continuous.
- (ii) T is continuous at 0.
- (iii) For all $q \in Q$ there exist $n \in \mathbb{N}, p_1, \ldots, p_n \in \mathcal{P}$ and $C \geqslant 0$ such that

$$q(Tx) \leqslant C \max_{1 \leqslant k \leqslant n} p_k(x)$$
 for all $x \in X$.

Dual space. Let (X, \mathcal{P}) be a locally convex space. The *dual space* X^* of X is the space of all linear functionals on X that are continuous with respect to the locally convex topology of X.

Lemma 10. Let f be a linear functional on a locally convex space X. Then $f \in X^*$ if and only if ker f is closed.

Theorem 11. (Hahn–Banach) Let (X, \mathcal{P}) be a locally convex space.

- (i) Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ with $f|_{Y} = g$.
- (ii) Given a closed subspace Y of X and $x_0 \in X \setminus Y$, there exists $f \in X^*$ such that $f \upharpoonright_Y = 0$ and $f(x_0) \neq 0$.

Remark. It follows that X^* separates the points of X.

The dual space of L_p . We fix a measure space $(\Omega, \mathcal{F}, \mu)$. Let $1 \leq p < \infty$ and let q be the conjugate index of p (i.e., $1 < q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$). For $g \in L_q = L_q(\mu)$ define

$$\varphi_g \colon L_p \to \text{scalars} \ , \qquad \varphi_g(f) = \int_{\Omega} f g \, \mathrm{d} \mu \ .$$

By Hölder's inequality we have φ_g is well-defined and $|\varphi_g(f)| \leq ||f||_p \cdot ||g||_q$. By linearity of integration, φ_g is linear, and hence $\varphi_g \in L_p^*$ with $||\varphi_g|| \leq ||g||_q$. We have thus obtained a function

$$\varphi \colon L_q \to L_p^* \ , \quad g \mapsto \varphi_g \ .$$

This is linear by linearity of integration and bounded with $\|\varphi\| \leq 1$.

Theorem 12. Let $(\Omega, \mathcal{F}, \mu)$, p, q, φ be as above.

- (i) If $1 , then <math>\varphi$ is an isometric isomorphism. Thus $L_p^* \cong L_q$.
- (ii) If p=1 and in addition μ is σ -finite, then φ is an isometric isomorphism. Thus, in this case, we have $L_1^* \cong L_{\infty}$.

Remarks. 1. Recall that μ is σ -finite if there is a sequence (A_n) in \mathcal{F} such that $\Omega = \bigcup A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.

2. One approach for proving surjectivity of φ is via the Radon–Nikodym theorem. We shall follow a different path via uniform convexity in Chapter 4.

Complex measures. Let Ω be a set and \mathcal{F} be a σ -field on Ω . A complex measure on \mathcal{F} is a countably additive set function $\nu \colon \mathcal{F} \to \mathbb{C}$. The total variation measure $|\nu|$ of ν is defined at $A \in \mathcal{F}$ as follows.

$$|\nu|(A) = \sup \left\{ \sum_{k=1}^{n} |\nu(A_k)| : A = \bigcup_{k=1}^{n} A_k \text{ is a measurable partition of } A \right\}.$$

(Measurable partition means that $A_k \in \mathcal{F}$ for all k, and $A_j \cap A_k = \emptyset$ for all $j \neq k$.) It is easy to check that $|\nu| \colon \mathcal{F} \to [0, \infty]$ is a positive measure. (The expression positive measure simply means measure but is used for emphasis to distinguish it from complex and signed measures.) It is also straightforward to verify that $|\nu|$ is the smallest positive measure that dominates ν . It is in fact a finite measure, i.e., that $|\nu|(\Omega) < \infty$. We define the total variation $||\nu||_1$ of ν by $||\nu||_1 = |\nu|(\Omega)$.

Signed measures. Let (Ω, \mathcal{F}) be as before. A *signed measure on* \mathcal{F} is a countably additive set function $\nu \colon \mathcal{F} \to \mathbb{R}$. Every signed measure is in particular a complex measure.

Theorem 13. Let $\nu \colon \mathcal{F} \to \mathbb{R}$ be a signed measure on the σ -field \mathcal{F} . Then there exist unique finite positive measures ν^+ and ν^- satisfying $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$.

Remarks. 1. This decomposition $\nu = \nu^+ - \nu^-$ of ν is called the *Jordan decomposition* of ν .

2. Let $\nu \colon \mathcal{F} \to \mathbb{C}$ be a complex measure. Then $\operatorname{Re}(\nu)$ and $\operatorname{Im}(\nu)$ are signed measures, so they have Jordan decompositions $\nu_1 - \nu_2$ and $\nu_3 - \nu_4$, respectively. We then obtain the expression $\nu = \nu_1 - \nu_2 + \mathrm{i}\nu_3 - \mathrm{i}\nu_4$ called the Jordan decomposition of ν . It follows that $\nu_k \leqslant |\nu|$ for all k, and $|\nu| \leqslant \nu_1 + \nu_2 + \nu_3 + \nu_4$. From this we deduce that $|\nu|$ is a finite measure.

Integration with respect to complex measures. Let Ω be a set, \mathcal{F} be a σ -field on Ω , and ν be a complex measure on \mathcal{F} . A measurable function $f:\Omega\to\mathbb{C}$ is ν -integrable if $\int_{\Omega} |f| \, \mathrm{d}|\nu| < \infty$. In that case we define

$$\int_{\Omega} f \, d\nu = \int_{\Omega} f \, d\nu_1 - \int_{\Omega} f \, d\nu_2 + i \int_{\Omega} f \, d\nu_3 - i \int_{\Omega} f \, d\nu_4 ,$$

where $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ is the Jordan decomposition of ν . Note that f is ν -integrable if and only if f is ν_k -integrable for each k. The following properties are easy to check:

- 1. $\int_{\Omega} \mathbf{1}_A d\nu = \nu(A)$ for all $A \in \mathcal{F}$.
- Linearity: given ν-integrable functions f, g and complex numbers a, b, the function af + bg is ν-integrable, and ∫_Ω(af + bg) dν = a ∫_Ω f dν + b ∫_Ω g dν.
 Dominated convergence (D.C.): let f_n, n ∈ N, be measurable functions that
- **3.** Dominated convergence (D.C.): let f_n , $n \in \mathbb{N}$, be measurable functions that converge a.e. to a measurable function f. Assume that there exists $g \in L_1(|\nu|)$ with $|f_n| \leq g$ for all n. Then f_n , f are ν -integrable and $\int_{\Omega} f_n \, d\nu \to \int_{\Omega} f \, d\nu$.
- **4.** $\left| \int_{\Omega} f \, d\nu \right| \leq \int_{\Omega} |f| \, d|\nu|$ for all ν -integrable f.

C(K) spaces. We fix a compact Hausdorff space K. We shall be interested in the following spaces and sets.

$$C(K) = \{ f \colon K \to \mathbb{C} : f \text{ is continuous} \}$$

$$C^{\mathbb{R}}(K) = \{ f \colon K \to \mathbb{R} : f \text{ is continuous} \}$$

$$C^+(K) = \{ f \in C(K) : f \geqslant 0 \text{ on } K \}$$

$$M(K) = C(K)^* = \{ \varphi \colon C(K) \to \mathbb{C} : \varphi \text{ is linear and continuous} \}$$

$$M^{\mathbb{R}}(K) = \{ \varphi \in M(K) : \varphi(f) \in \mathbb{R} \text{ for all } f \in C^{\mathbb{R}}(K) \}$$

$$M^+(K) = \{ \varphi \colon C(K) \to \mathbb{C} : \varphi \text{ is linear and } \varphi(f) \geqslant 0 \text{ for all } f \in C^+(K) \}$$

Note. 1. $M^{\mathbb{R}}(K)$ is a closed, real-linear subspace of M(K). $C^{\mathbb{R}}(K)^* \cong M^{\mathbb{R}}(K)$. $f \mapsto f \upharpoonright_{C^{\mathbb{R}}(K)} : M^{\mathbb{R}}(K) \to C^{\mathbb{R}}(K)^*$ is an isometric real-linear isometry.

2. Elements of $M^+(K)$ are called *positive linear functionals*. These are automatically continuous, and in fact

$$M^+(K) = \{ \varphi \in M(K) : ||\varphi|| = \varphi(\mathbf{1}_K) \}.$$

Borel measures and regularity. Let X be a Hausdorff topological space and \mathcal{G} be the collection of open subsets of X. The Borel σ -field on X is defined to be $\mathcal{B} = \sigma(\mathcal{G})$, the σ -field generated by \mathcal{G} , i.e., the smallest σ -field on X containing \mathcal{G} . Equivalently, \mathcal{B} is the intersection of all σ -fields on X that contain \mathcal{G} . Members of \mathcal{B} are called Borel sets.

A Borel measure on X is a measure on \mathcal{B} . Given a Borel measure μ on X, we say μ is regular if the following hold:

- (i) $\mu(E) < \infty$ for all compact $E \subset X$;
- (ii) $\mu(A) = \inf \{ \mu(U) : A \subset U \in \mathcal{G} \}$ for all $A \in \mathcal{B}$;
- (iii) $\mu(U) = \sup \{ \mu(E) : E \subset U, E \text{ compact} \} \text{ for all } U \in \mathcal{G}.$

A complex Borel measure ν on X is defined to be regular if $|\nu|$ is a regular measure on X.

Note that if X is compact Hausdorff, then a Borel measure μ on X is regular if and only if

$$\mu(X) < \infty$$
 and $\mu(A) = \inf \{ \mu(U) : A \subset U \in \mathcal{G} \} \quad \forall A \in \mathcal{B}$

which in turn is equivalent to

$$\mu(X) < \infty$$
 and $\mu(A) = \sup{\{\mu(E) : E \subset A, E \text{ closed}\}} \quad \forall A \in \mathcal{B}$.

Example. Lebesgue measure on \mathbb{R} is a regular Borel measure.

The dual space of C(K). Let ν be a complex Borel measure on K. For $f \in C(K)$ we have

$$\int_{K} |f| \, \mathrm{d}|\nu| \leqslant ||f||_{\infty} \cdot |\nu|(K) ,$$

and hence f is ν -integrable. The function $\varphi \colon C(K) \to \mathbb{C}$ given by $\varphi(f) = \int_K f \, \mathrm{d}\nu$ is linear and bounded with $\|\varphi\| \leqslant \|\nu\|_1$. Thus $\varphi \in M(K)$. Note that if ν is a signed measure, then $\varphi \in M^{\mathbb{R}}(K)$, and if ν is a positive measure, then $\varphi \in M^+(K)$.

Theorem 14. (Riesz Representation Theorem) For every $\varphi \in M(K)$ there is a unique regular complex Borel measure ν on K that represents φ :

$$\varphi(f) = \int_K f \, \mathrm{d}\nu$$
 for all $f \in C(K)$.

Moreover, we have $\|\varphi\| = \|\nu\|_1$. If $\varphi \in M^{\mathbb{R}}(K)$ then ν is a signed measure, and if $\varphi \in M^+(K)$, then ν is a positive measure.

Corollary 15. The space of regular complex Borel measures on K is a complex Banach space in the total variation norm, and it is isometrically isomorphic to $M(K) = C(K)^*$.

The space of regular signed Borel measures on K is a real Banach space in the total variation norm, and it is isometrically isomorphic to $M^{\mathbb{R}}(K) \cong C^{\mathbb{R}}(K)^*$.

2 Weak topologies

Let X be a set and \mathcal{F} be a family of functions such that each $f \in \mathcal{F}$ is a function $f \colon X \to Y_f$ where Y_f is a topological space. The weak topology $\sigma(X, \mathcal{F})$ of X is the smallest topology on X with respect to which every $f \in \mathcal{F}$ is continuous.

Remarks. 1. The family $S = \{f^{-1}(U) : f \in \mathcal{F}, U \subset Y_f \text{ an open set} \}$ is a sub-base for $\sigma(X, \mathcal{F})$. This means that $\sigma(X, \mathcal{F})$ is the topology generated by S, *i.e.*, the smallest topology containing S. Equivalently, the family of finite intersections of members of S is a base for $\sigma(X, \mathcal{F})$, *i.e.*, $\sigma(X, \mathcal{F})$ consists of arbitrary unions of finite intersections of members of S. To spell this out, a set $V \subset X$ is open in the weak topology $\sigma(X, \mathcal{F})$ if and only if for all $x \in V$ there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathcal{F}$ and open sets $U_j \subset Y_{f_j}$, $1 \leq j \leq n$, such that

$$x \in \bigcap_{j=1}^{n} f_j^{-1}(U_j) \subset V$$
.

To put it in another way, $V \subset X$ is open in the weak topology if and only if for all $x \in V$ there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathcal{F}$ and open neighbourhoods U_j of $f_j(x)$ in Y_{f_j} , $1 \leq j \leq n$, such that $\bigcap_{j=1}^n f_j^{-1}(U_j) \subset V$.

2. Suppose that S_f is a sub-base for the topology of Y_f for each $f \in \mathcal{F}$. Then $\{f^{-1}(U): f \in \mathcal{F}, U \in S_f\}$ is a sub-base for $\sigma(X, \mathcal{F})$.

3. If Y_f is a Hausdorff space for each $f \in \mathcal{F}$ and \mathcal{F} separates the points of X (for all $x \neq y$ in X there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$), then $\sigma(X, \mathcal{F})$ is Hausdorff.

4. Let $Y \subset X$ and set $\mathcal{F}_Y = \{f |_Y : f \in \mathcal{F}\}$. Then $\sigma(X, \mathcal{F})|_Y = \sigma(Y, \mathcal{F}_Y)$, *i.e.*, the subspace topology on Y induced by the weak topology $\sigma(X, \mathcal{F})$ of X is the same as the weak topology on Y defined by \mathcal{F}_Y .

5. (Universal property) Let Z be a topological space and $g: Z \to X$ a function. Then g is continuous if and only if $f \circ g: Z \to Y_f$ is continuous for every $f \in \mathcal{F}$. This universal property characterizes the weak topology (cf. Examples Sheet 2).

Examples. 1. Let X be a topological space, $Y \subset X$ and $\iota \colon Y \to X$ be the inclusion map. Then the weak topology $\sigma(Y, \{\iota\})$ is the subspace topology of Y induced by X.

2. Let Γ be an arbitrary set, and let X_{γ} be a topological space for each $\gamma \in \Gamma$. Let X be the Cartesian product $\prod_{\gamma \in \Gamma} X_{\gamma}$. Thus

 $X = \{x : x \text{ is a function with domain } \Gamma, \text{ and } x(\gamma) \in X_{\gamma} \text{ for all } \gamma \in \Gamma\}$.

For $x \in X$ we often write x_{γ} instead of $x(\gamma)$, and think of x as the " Γ -tuple" $(x_{\gamma})_{\gamma \in \Gamma}$. For each γ we consider the function $\pi_{\gamma} \colon X \to X_{\gamma}$ given by $x \mapsto x(\gamma)$ (or $(x_{\delta})_{\delta \in \Gamma} \mapsto x_{\gamma}$) called evaluation at γ or projection onto X_{γ} . The product topology on X is the weak topology $\sigma(X, \{\pi_{\gamma} : \gamma \in \Gamma\})$. Note that $V \subset X$ is open if and only if for all $x = (x_{\gamma})_{\gamma \in \Gamma} \in V$ there exist $n \in \mathbb{N}, \gamma_1, \ldots, \gamma_n \in \Gamma$ and open neighbourhoods U_j of x_{γ_j} in X_{γ_j} such that

$$\{y = (y_{\gamma})_{\gamma \in \Gamma} : y_{\gamma_j} \in U_j \text{ for } 1 \leqslant j \leqslant n\} \subset V$$
.

Proposition 1. Assume that X is a set, and for each $n \in \mathbb{N}$ we are given a metric space (Y_n, d_n) and a function $f_n \colon X \to Y_n$. Further assume that $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ separates the points of X. Then the weak topology $\sigma(X, \mathcal{F})$ of X is metrizable.

Remark. Without the assumption that \mathcal{F} separates the points of X, we can conclude that $\sigma(X, \mathcal{F})$ is pseudo-metrizable.

Theorem 2. (Tychonov) The product of compact topological spaces is compact in the product topology.

Weak topologies on vector spaces. Let E be a real or complex vector space. Let F be a subspace of the space of all linear functionals on E that separates the points of E (for all $x \in E$, $x \neq 0$, there exists $f \in F$ such that $f(x) \neq 0$). Consider the weak topology $\sigma(E, F)$ on E. Note that $U \subset E$ is open if and only if for all $x \in U$ there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in F$ and $\varepsilon > 0$ such that

$$\{y \in E : |f_i(y-x)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \subset U$$
.

For $f \in F$ define $p_f \colon E \to \mathbb{R}$ by $p_f(x) = |f(x)|$. Set $\mathcal{P} = \{p_f \colon f \in F\}$. Then \mathcal{P} is a family of seminorms on E that separates the points of E, and the topology of the locally convex space (E, \mathcal{P}) is precisely $\sigma(E, F)$. In particular, $\sigma(E, F)$ is a Hausdorff topology with respect to which addition and scalar multiplication are continuous.

Lemma 3. Let E be as above. Let f, g_1, g_2, \ldots, g_n be linear functionals on E. If $\bigcap_{j=1}^n \ker g_j \subset \ker f$, then $f \in \operatorname{span}\{g_1, \ldots, g_n\}$.

Proposition 4. Let E and F be as above. Then a linear functional f on E is $\sigma(E, F)$ -continuous if and only if $f \in F$. In other words $(E, \sigma(E, F))^* = F$.

Note. Recall that $(E, \sigma(E, F))^*$ denotes the dual space of the locally convex space $(E, \sigma(E, F))$, *i.e.*, the space of linear functionals that are continuous with respect to $\sigma(E, F)$.

Examples. The following two examples of weak topologies on vector spaces are the central objects of interest in this chapter.

1. Let X be a normed space. The weak topology on X is the weak topology $\sigma(X, X^*)$ on X. Note that X^* separates the points of X by Hahn–Banach. We shall sometimes denote X with the weak topology by (X, w). Open sets in the weak topology are called weakly open or w-open. Note that $U \subset X$ is w-open if and only if for all $x \in U$ there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in X^*$ and $\varepsilon > 0$ such that

$$\{y \in X : |f_i(y-x)| < \varepsilon \text{ for } 1 \le i \le n\} \subset U$$
.

2. Let X be a normed space. The weak-star topology (or w^* -topology) on X^* is the weak topology $\sigma(X^*,X)$ on X^* . Here we identify X with its image in X^{**} under the canonical embedding. We shall sometimes denote X^* with the w^* -topology by (X^*,w^*) . Open sets in the weak-star topology are called w^* -open. Note that $U \subset X^*$ is w^* -open if and only if for all $f \in U$ there exist $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that

$$\{g \in X^* : |g(x_i) - f(x_i)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \subset U.$$

Properties. 1. (X, w) and (X^*, w^*) are locally convex spaces, so in particular they are Hausdorff, and addition and scalar multiplication are continuous.

- **2.** $\sigma(X, X^*) \subset \|\cdot\|$ -topology with equality if and only if dim $X < \infty$.
- **3.** $\sigma(X^*,X) \subset \sigma(X^*,X^{**}) \subset \|\cdot\|$ -topology with equality in the second inclusion if and only if dim $X < \infty$, and with equality in the first inclusion if and only if X is reflexive (*cf.* Proposition 5 below).
- **4.** If Y is a subspace of X, then $\sigma(X, X^*) \upharpoonright_Y = \sigma(Y, Y^*)$. This follows from Remark 4 on page 10 and from the Hahn–Banach theorem.

Similarly, we have $\sigma(X^{**}, X^*) \upharpoonright_X = \sigma(X, X^*)$, *i.e.*, the subspace topology on X induced by the w^* -topology of X^{**} is the weak topology of X. Thus, the canonical embedding $X \to X^{**}$ is a w-to- w^* -homeomorphism from X onto its image (as well as being an isometric isomorphism).

Proposition 5. Let X be a normed space. Then

- (i) a linear functional f on X is weakly continuous (*i.e.*, continuous with respect to the weak topology on X) if and only if $f \in X^*$. Briefly, $(X, w)^* = X^*$;
- (ii) a linear functional φ on X^* is w^* -continuous (i.e., continuous with respect to the weak-star topology on X^*) if and only if $\varphi \in X$, i.e., there exists $x \in X$ with $\varphi = \hat{x}$. Briefly, $(X^*, w^*)^* = X$.

It follows that on X^* the weak and weak-star topologies coincide, *i.e.*, we have $\sigma(X^*, X^{**}) = \sigma(X^*, X)$ if and only if X is reflexive.

Definition. Let X be a normed space.

A set $A \subset X$ is weakly bounded if $\{f(x) : x \in A\}$ is bounded for all $f \in X^*$. Equivalently, for each w-open ngbd U of 0, there exists $\lambda > 0$ such that $A \subset \lambda U$. A set $B \subset X^*$ is weak-star bounded if $\{f(x) : f \in B\}$ is bounded for all $x \in X$. Equivalently, for each w^* -open ngbd U of 0, there exists $\lambda > 0$ such that $B \subset \lambda U$.

Principal of Uniform Boundedness (PUB). If X is a Banach space, Y is a normed space, $\mathcal{T} \subset \mathcal{B}(X,Y)$ is pointwise bounded ($\sup\{\|Tx\|: T \in \mathcal{T}\} < \infty$ for all $x \in X$), then \mathcal{T} is uniformly bounded ($\sup\{\|T\|: T \in \mathcal{T}\} < \infty$).

Proposition 6. Let X be a normed space.

- (i) Any weakly bounded set in X is bounded in norm.
- (ii) If X is complete then any weak-star bounded set in X^* is bounded in norm.

Notation. 1. We write $x_n \xrightarrow{w} x$ and say x_n converges weakly to x if $x_n \to x$ in the weak topology (in some normed space X). This happens if and only if $\langle x_n, f \rangle \to \langle x, f \rangle$ for all $f \in X^*$.

2. We write $f_n \xrightarrow{w^*} f$ and say f_n converges weak-star to f if $f_n \to f$ in the weak-star topology (in some dual space X^*). This happens if and only if $\langle x, f_n \rangle \to \langle x, f \rangle$ for all $x \in X$.

A consequence of PUB. Let X be a Banach space, Y a normed space, and (T_n) a sequence in $\mathcal{B}(X,Y)$ that converges pointwise to a function T. Then $T \in \mathcal{B}(X,Y)$, $\sup ||T_n|| < \infty$ and $||T|| \leq \liminf ||T_n||$.

Proposition 7. Let X be a normed space.

- (i) If $x_n \xrightarrow{w} x$ in X, then $\sup ||x_n|| < \infty$ and $||x|| \le \liminf ||x_n||$.
- (ii) If $f_n \xrightarrow{w^*} f$ in X^* and X is complete, then $\sup ||f_n|| < \infty$ and $||f|| \le \liminf ||f_n||$.

The Hahn–Banach Separation Theorems. Let (X, \mathcal{P}) be a locally convex space. Let C be a convex subset of X with $0 \in \text{int } C$. Define

$$\mu_C: X \to \mathbb{R}$$
, $\mu_C(x) = \inf\{t > 0 : x \in tC\}$.

This function is well-defined and is called the Minkowski functional or gauge functional of C.

Example. If X is a normed space and $C = B_X$, then $\mu_C = \|\cdot\|$.

Lemma 8. The function μ_C is a positive homogeneous, subadditive functional. Moreover, we have

$$\{x \in X : \mu_C(x) < 1\} \subset C \subset \{x \in X : \mu_C(x) \le 1\}$$

with equality in the first inclusion when C is open and equality in the second inclusion when C is closed.

Remark. If C is symmetric in the case of real scalars, or balanced in the case of complex scalars, then μ_C is a seminorm. If in addition C is bounded, then μ_C is a norm.

Theorem 9. (Hahn–Banach separation theorem) Let (X, \mathcal{P}) be a locally convex space and C be an open convex subset of X with $0 \in C$. Let $x_0 \in X \setminus C$. Then there exists $f \in X^*$ such that $f(x_0) > f(x)$ for all $x \in C$ if the scalar field is \mathbb{R} , and $\text{Re} f(x_0) > \text{Re} f(x)$ for all $x \in C$ if the scalar field is \mathbb{C} .

Remark. From now on, for the rest of this chapter, we assume that the scalar field is \mathbb{R} . It is straightforward to modify the statements of theorems, their proofs, etc, in the case of complex scalars.

Theorem 10. (Hahn–Banach separation theorem) Let (X, \mathcal{P}) be a locally convex space and A, B be non-empty disjoint convex subsets of X.

- (i) If A is open, then there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(x) < \alpha \le f(y)$ for all $x \in A, y \in B$.
- (ii) If A is compact and B is closed, then there exists $f \in X^*$ such that $\sup_A f < \inf_B f$.

Theorem 11. (Mazur) Let C be a convex subset of a normed space X. Then $\overline{C}^{\|\cdot\|} = \overline{C}^w$, *i.e.*, the norm-closure and weak-closure of C are the same. It follows that C is norm-closed if and only if C is weakly closed.

Corollary 12. Assume that $x_n \xrightarrow{w} 0$ in a normed space X. Then for all $\varepsilon > 0$ there exists $x \in \text{conv}\{x_n : n \in \mathbb{N}\}$ with $||x|| < \varepsilon$.

Remark. It follows that there exist $p_1 < q_1 < p_2 < q_2 < \dots$ and convex combinations $\sum_{i=p_n}^{q_n} t_i x_i$ that converge to zero in norm. In some but not all cases a stronger conclusion can be obtained: see Examples Sheet 2, Question 13.

Theorem 13. (Banach–Alaoglu) The dual ball B_{X^*} is w^* -compact for any normed space X.

Proposition 14. Let X be a normed space and K be a compact Hausdorff space.

- (i) X is separable if and only if (B_{X^*}, w^*) is metrizable.
- (ii) C(K) is separable if and only if K is metrizable.

Remarks. 1. If X is separable, then (B_{X^*}, w^*) is a compact metric space. In particular, B_{X^*} is w^* -sequentially compact.

- **2.** If X is separable, then X^* is w^* -separable. It is an easy consequence of Mazur's theorem that X is separable if and only if X is weakly separable. Thus, the previous statement reads: if X is w-separable, then X^* is w^* -separable. The converse is false, e.g., for $X = \ell_{\infty}$ (see the remark following Goldstine's theorem).
- **3.** The proof shows that any compact Hausdorff space K is a subspace of $(B_{C(K)^*}, w^*)$.
- **4.** The proof shows that every normed space X embeds isometrically into C(K) for some compact Hausdorff space. In particular, this holds with $K = (B_{X^*}, w^*)$ (cf. Theorem 19 below).

Proposition 15. X^* is separable if and only if (B_X, w) is metrizable.

Theorem 16. (Goldstine) $\overline{B_X}^{w^*} = B_{X^{**}}$, *i.e.*, the w^* -closure of the unit ball B_X of a normed space X in the second dual X^{**} is $B_{X^{**}}$.

Remark. It follows from Goldstine that $\overline{X}^{w^*} = X^{**}$. Thus, if X is separable, then X^{**} is w^* -separable. For example, ℓ_{∞}^* is w^* -separable.

Theorem 17. Let X be a Banach space. Then TFAE.

- (i) X is reflexive.
- (ii) (B_X, w) is compact.
- (iii) X^* is reflexive.

Remark. It follows that if X is separable and reflexive, then (B_X, w) is a compact metric space, and hence sequentially compact.

Lemma 18. For every non-empty compact metric space K there is a continuous surjection $\varphi \colon \Delta \to K$, where $\Delta = \{0,1\}^{\mathbb{N}}$ with the product topology.

Note. Δ is homeomorphic via the map $(\varepsilon_i) \mapsto \sum_i (2\varepsilon_i)3^{-i}$ to the middle-third Cantor set.

Theorem 19. Every separable Banach space embeds isometrically into C[0,1]. Thus the separable space C[0,1] is isometrically universal for the class of separable Banach spaces.

3 Schauder bases

Definition. A Schauder basis or simply basis of a Banach space X is a sequence (e_n) in X such that for all $x \in X$ there exists a unique sequence (a_n) of scalars such that x is the norm-convergent sum $x = \sum_{n=1}^{\infty} a_n e_n$.

Notation. For a sequence (e_n) in X, we let $[e_n] = \overline{\operatorname{span}}\{e_n : n \in \mathbb{N}\}.$

Note. If (e_n) is a basis of X, then (e_n) is linearly independent and $X = [e_n]$. Thus, X is separable and infinite-dimensional.

Definition. A sequence (e_n) in a Banach space X is called a *basic sequence* if it is a basis of its closed linear span $[e_n]$.

Theorem 1. Let (e_n) be a sequence in a Banach space X. Then (e_n) is a basis of X if and only if the following three conditions hold:

- (i) $e_n \neq 0$ for all $n \in \mathbb{N}$.
- (ii) There exists $C \ge 1$ such that

$$\left\| \sum_{i=1}^{m} a_i e_i \right\| \leqslant C \left\| \sum_{i=1}^{n} a_i e_i \right\|$$

for all scalar sequences (a_i) and all $1 \leq m \leq n$ in \mathbb{N} .

(iii)
$$X = [e_n].$$

It follows that (e_n) is a basic sequence if and only if conditions (i) and (ii) hold.

Remark. If (e_n) is a basis of X, then the projections $P_n \colon X \to X$ defined by $P_n\left(\sum_{i=1}^{\infty} a_i e_i\right) = \sum_{i=1}^{n} a_i e_i$ are called the *basis projections*. By the above theorem, $\{P_n : n \in \mathbb{N}\}$ is uniformly bounded. The constant $C = \sup_{n \in \mathbb{N}} ||P_n||$ is called the *basis constant* of (e_n) , and it is the least C that satisfies condition (ii) above. A basis with basis constant 1 is called a *monotone basis*.

Examples. 1. The *Schauder system* is a monotone basis of C[0,1].

- **2.** The unit vector basis is a monotone basis of ℓ_p , $1 \leq p < \infty$, and of c_0 .
- **3.** The *Haar system* is a monotone basis of $L_p[0,1]$ for $1 \le p < \infty$.

Remark. A famous result of Per Enflo shows that there are separable infinite-dimensional Banach spaces without bases. However, we have the following.

Theorem 2. (Mazur) Every infinite-dimensional Banach space contains a basic sequence. More precisely, for any $\varepsilon > 0$, every infinite-dimensional Banach space contains a normalised basic sequence with basis constant at most $1 + \varepsilon$.

Lemma 3. Let F be a finite-dimensional subspace of an infinite-dimensional Banach space X, and let $\varepsilon > 0$. Then there exists $y \in S_X$ such that $||x|| \le (1+\varepsilon)||x+\lambda y||$ for all $x \in F$ and all scalars λ .

Remark. The proof of the lemma shows that there is a finite-codimensional subspace Y of X such that any $y \in S_Y$ works.

Dual bases. Let (e_n) be a basis of a Banach space X, let P_n , $n \in \mathbb{N}$, be the corresponding basis projections, and let C be the basis constant. For $n \in \mathbb{N}$, define the n^{th} coordinate functional e_n^* by

$$\left\langle \sum a_i e_i, e_n^* \right\rangle = a_n .$$

Then $e_n^* \in X^*$ with $\frac{1}{\|e_n\|} \leqslant \|e_n^*\| \leqslant \frac{2C}{\|e_n\|}$. Moreover,

$$P_n^*(x^*) = \sum_{i=1}^n x^*(e_i)e_i^* \qquad (x^* \in X^*, \ n \in \mathbb{N})$$

It follows that (e_i^*) is a basic sequence in X^* whose n^{th} basis projection is the restriction to $[e_i^*]$ of P_n^* , and hence its basis constant is at most C. We call (e_i^*) the dual basis or biorthogonal sequence of (e_i) .

Equivalence of bases and perturbation. We begin with a simple result.

Proposition 4. For two basic sequences (e_n) and (f_n) in possibly different Banach spaces the following conditions are equivalent.

- (i) $\sum a_i e_i$ converges if and only if $\sum a_i f_i$ converges for all (a_i) .
- (ii) The map $T(e_n) = f_n$ extends to an isomorphism $T: [e_n] \to [f_n]$.
- (iii) There exist constants A > 0 and B > 0 such that

$$A \left\| \sum a_i e_i \right\| \le \left\| \sum a_i f_i \right\| \le B \left\| \sum a_i e_i \right\|$$

for all $(a_i) \in c_{00}$.

Definition. Two basic sequences (e_n) and (f_n) in possibly different Banach spaces are *equivalent*, written $(e_n) \sim (f_n)$, if any of the conditions (i), (ii) or (iii) in the proposition above is satisfied. More precisely, we say that (e_n) and (f_n) are C-equivalent, written $(e_n) \sim (f_n)$, if (iii) holds with $A/B \leq C$, or equivalently, if (ii) holds with $||T||||T^{-1}|| \leq C$.

Proposition 5. Let (e_n) be a basic sequence in a Banach space X with dual basis (e_n^*) . If a sequence (f_n) in X satisfies

(1)
$$\sum_{n=1}^{\infty} ||e_n^*|| ||e_n - f_n|| < 1$$

then (f_n) is a basic sequence which is C-equivalent to (e_n) , where $C = \frac{1+\gamma}{1-\gamma}$ and $\gamma = \sum_n ||e_n^*|| ||e_n - f_n||$. Moreover, if $[e_n]$ is complemented in X, then so is $[f_n]$.

Block bases. Let (e_n) be a basis of a Banach space X with basis constant C. For $x = \sum_n a_n e_n \in X$, we define the *support* of X (with respect to (e_n)) to be the set

$$supp(x) = \{ n \in \mathbb{N} : a_n \neq 0 \} .$$

Note that $\operatorname{supp}(x) = \{n \in \mathbb{N} : e_n^*(x) \neq 0\}$ where (e_n^*) is the dual basis.

For subsets A, B of \mathbb{N} , we write A < B if a < b for all $a \in A$ and $b \in B$. For vectors $x, y \in X$, we write x < y if $\operatorname{supp}(x) < \operatorname{supp}(y)$. A block basis of (e_n) is a sequence $u_1 < u_2 < \ldots$ of non-zero vectors in X. Note that (u_n) is then a basic sequence with basis constant at most C.

A sequence (x_n) in a Banach space X is *seminormalised* if there exist constants a > 0 and b > 0 such that $a \leq ||x_n|| \leq b$ for all $n \in \mathbb{N}$.

Proposition 6. Let (e_n) be a basis of a Banach space X with basis projections P_n , $n \in \mathbb{N}$. Let (x_n) be a seminormalised sequence in X such that $P_k(x_n) \to 0$ as $n \to \infty$ for each $k \in \mathbb{N}$. Then (x_n) has a basic subsequence equivalent to some block basis of (e_n) .