Winding numbers, Cauchy's Theorem and Cauchy's Integral Formula

We recall a number of definitions and results from complex analysis. Different versions of these exist, so the purpose here is to spell out the ones used in lectures.

1. Paths

A **path** is a continuously differentiable function $\gamma \colon [a,b] \to \mathbb{C}$ defined on some closed bounded interval of \mathbb{R} .

The **track** of γ is $[\gamma] = {\gamma(t) : t \in [a, b]}$ (i.e., simply the image of γ).

Given $S \subset \mathbb{C}$, we say γ is a **path in** S if $[\gamma] \subset S$.

The **length** of γ is $\ell(\gamma) = \int_a^b |\gamma'(t)| dt$.

Integral along γ : $\int_{\gamma} f(z) dz = \int_{a}^{b} f(\gamma(t)) \gamma'(t) dt$ for $f: [\gamma] \to \mathbb{C}$ continuous.

2. Chains and cycles

A chain is a finite sequence $\Gamma = (\gamma_1, \dots, \gamma_n)$ of paths $\gamma_i : [a_i, b_i] \to \mathbb{C}$.

The **track** of
$$\Gamma$$
 is $[\Gamma] = \bigcup_{j=1}^{n} [\gamma_j]$.

Given $S \subset \mathbb{C}$, we say Γ is a **chain in** S if $[\Gamma] \subset S$.

The **length** of
$$\Gamma$$
 is $\ell(\Gamma) = \sum_{j=1}^{n} \ell(\gamma_j)$.

Integral along
$$\Gamma$$
: $\int_{\Gamma} f(z) dz = \sum_{j=1}^{n} \int_{\gamma_{j}} f(z) dz$ for $f: [\Gamma] \to \mathbb{C}$ continuous.

 Γ is a **cycle** if there is a permutation $\rho \in S_n$ such that $\gamma_j(b_j) = \gamma_{\rho(j)}(a_{\rho(j)})$ for $j = 1, \ldots, n$.

3. Winding numbers

Let Γ be a cycle and $w \in \mathbb{C} \setminus [\Gamma]$. The winding number of Γ round w is

$$n(\Gamma, w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\mathrm{d}z}{z - w}$$

which is always an integer.

4. Lemma

Assume we have $\emptyset \neq K \subset U \subset \mathbb{C}$, where K is compact and U is open. Then there is a cycle Γ in $U \setminus K$ such that

$$n(\Gamma, w) = \begin{cases} 1 & \text{if } w \in K \\ 0 & \text{if } w \notin U \end{cases}$$

5. Cauchy's Theorem

Let U be a non-empty open subset of \mathbb{C} , and let Γ be a cycle in U such that $n(\Gamma, w) = 0$ for all $w \notin U$ (Γ does not wind round any point outside U). Then

$$\int_{\Gamma} f(z) \, \mathrm{d}z = 0$$

for any holomorphic function $f: U \to \mathbb{C}$.

6. Cauchy's Integral Formula

Let U be a non-empty open subset of \mathbb{C} , and let Γ be a cycle in U such that $n(\Gamma, w) = 0$ for all $w \notin U$ (Γ does not wind round any point outside U). Let $f: U \to \mathbb{C}$ be holomorphic. Then

$$n(\Gamma, a)f(a) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - a} dz$$

for any $a \in U \setminus [\Gamma]$.