1. Prove Bessel's inequality: if (e_n) is an orthonormal sequence in an inner product space, then $\sum_n |\langle x, e_n\rangle|^2 \leqslant \|x\|^2$ for every x.

2. Show that $T \in \mathcal{B}(H)$ is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.

3. Let $a = (a_n) \in \ell_\infty$. Define $T\colon \ell_2 \to \ell_2$ by $T(\sum x_n e_n) = \sum a_n x_n e_n$. Find the eigenvalues, approximate eigenvalues and the spectrum of T. Show that T is compact if and only if $(a_n) \in \mathrm{c}_0$.

4. Let U be a unitary operator on a complex Hilbert space. Show that $\sigma(U) \subset \mathbb{T}$.

5. Let H be a complex Hilbert space with orthonormal basis $(e_n)_{n=-\infty}^{\infty}$. The *bilateral shift* is the operator T on H defined by $Te_n = e_{n+1}$ $(n \in \mathbb{Z})$. Find the spectrum of T.

6. Let K be a non-empty compact subset of $\mathbb{C}$. Show that there is an operator on ℓ_2 whose spectrum is K.

7. Prove that if T is a normal operator then $\sigma(T) = \sigma_{\mathrm{ap}}(T)$.

8. Let T be a unitary operator on a Hilbert space H, and suppose that T acts on a closed subspace Y of H. Must the restriction of T to Y be unitary?

9. Let T be a positive compact operator on a Hilbert space H. Show that there is a unique positive operator S on H with $S^2 = T$.

10. Show that $\mathcal{F}(\ell_2)$ is dense in $\mathcal{K}(\ell_2)$. Is $\mathcal{G}(\ell_2)$ dense in $\mathcal{B}(\ell_2)$?

11. Let $A = (a_{ij})_{i,j=1}^{\infty}$ be a complex matrix whose rows and columns form a bounded set in ℓ_2. Must A be the matrix of a bounded linear operator on ℓ_2?

12. Construct a compact operator T on a (non-zero) Hilbert space H so that T has no eigenvalues.

13. Let T be an operator on a Banach space X with $\|T\| < 1$. Show that $I - T$ has a square root, *i.e.*, there exists $S \in \mathcal{B}(X)$ such that $S^2 = I - T$.

14^+. Let X be a separable infinite-dimensional Banach space. Suppose that there is a constant C such that every finite-dimensional subspace E of X satisfies $d(E, \ell_2^n) \leqslant C$, where $n = \dim E$. Prove that X is isomorphic to ℓ_2.