1. Let $a = (a_n) \in \ell_\infty$. Define $T\colon \ell_2 \to \ell_2$ by $T(\sum x_n e_n) = \sum a_n x_n e_n$. Prove that $T \in \mathcal{B}(\ell_2)$ and that $\|T\| = \|a\|_\infty$.

2. Let f be a linear functional on a normed space X. Prove that f is continuous if and only if $\ker f$ is closed.

3. Show that no two of the spaces $\ell_1, \ell_2, \ell_\infty, \mathrm{c}_0$ are isomorphic.

4. Assume that X is an infinite-dimensional normed space. Show that there is a sequence (x_n) in the unit ball of X with $\|x_m - x_n\| \geqslant 1$ whenever $m \neq n$. Is it possible to replace $\geqslant$ by $>$?

5. Let X, Y be normed spaces that are dense in Banach spaces $\widetilde{X}$, $\widetilde{Y}$, respectively. Let $T \in \mathcal{B}(X, Y)$. Show that T extends to a unique $\widetilde{T} \in \mathcal{B}(\widetilde{X}, \widetilde{Y})$ with $\|\widetilde{T}\| = \|T\|$. So we may regard $\mathcal{B}(X, Y)$ as a subspace of $\mathcal{B}(\widetilde{X}, \widetilde{Y})$. Is $\mathcal{B}(X, Y)$ dense in $\mathcal{B}(\widetilde{X}, \widetilde{Y})$? If T is surjective, must $\widetilde{T}$ be surjective? If T is injective, must $\widetilde{T}$ be injective?

6. Let X be a non-empty countable complete metric space. Show that X has an isolated point.

7. Let Y be a proper subspace of a Banach space X. Can Y be dense $\mathcal{G}_\delta$, *i.e.*, can Y be the intersection of a sequence of dense open sets in X?

8. Let $1 \leqslant p < q$. Consider the *subset* $Y = \ell_p$ of the Banach space $X = (\ell_q, \|\cdot\|_q)$. Show that Y is meagre in X.

9. Suppose that $T\colon X \to Y$ satisfies the conditions in the Open Mapping Lemma. Show that Y is complete.

10. Let X be a closed subspace of ℓ_1. Assume that every $y = (x_{2n}) \in \ell_1$ extends to a sequence $x = (x_n) \in X$. Show that there is a constant C such that x can always be chosen to satisfy $\|x\| \leqslant C\|y\|$.

11. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that for every $x > 0$ we have $f(nx) \to 0$ as $n \to \infty$. Show that $f(x) \to 0$ as $x \to \infty$.

12. Assume that X is a closed subspace of $\big(C[0,1], \|\cdot\|_\infty\big)$ such that every element of X is continuously differentiable. Show that X is finite-dimensional.

13. Let $f\colon [0,1] \to \mathbb{R}$ be a pointwise limit of a sequence of continuous functions. Show that f has a point of continuity.

$^+$14. Let X be a normed space that is homeomorphic to a complete metric space. Prove that X is complete.

$^+$15. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that for every $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ with $f^{(m)}(x) = 0$ for all $m \geqslant n$. Prove that f is a polynomial.