1. Let $1 \leqslant p < q < \infty$. Show that $\ell_p \subsetneq \ell_q \subsetneq \mathrm{c}_0$. Is $\bigcup_{p\in[1,\infty)} \ell_p = \mathrm{c}_0$?

2. (i) Prove carefully that $C[0,1]$ is incomplete in the L_1-norm $\|\cdot\|_1$.

(ii) Show that the space $C^1[0,1]$ is incomplete in the uniform norm $\|\cdot\|_\infty$ but complete in the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$.

3. Let $T\colon \ell_p^n \to \ell_q^n$ be the identity map of the underlying vector space $\mathbb{R}^n$. Compute the operator norm of T for all possible values of p and q.

4. Show carefully that $\ell_1^* \cong \ell_\infty$ and $\mathrm{c}_0^* \cong \ell_1$.

5. Show directly that the spaces ℓ_p, $1 \leqslant p \leqslant \infty$, and c_0 are complete.

6. Prove that $C[0,1]$ is separable and that ℓ_∞ is not separable.

7. Let X be a normed space. For $x \in X \setminus \{0\}$ write $\pi(x) = x/\|x\|$. Is it true that $\|\pi(x) - \pi(y)\| \leqslant \|x - y\|$ whenever $\|x\|,\ \|y\| \geqslant 1$?

8. Prove that a normed space X is a Banach space if and only if every series $\sum x_n$ in X with $\sum \|x_n\| < \infty$ is convergent.

9. Show that the spaces c_0 and c are isomorphic. Are they isometrically isomorphic?

10. Let Y and Z be dense subspaces of a normed space X. Is $Y \cap Z$ dense in X?

11. Give two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space X such that $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ are isomorphic.

12. Assume that the vector space X is the (algebraic) direct sum of subspaces Y and Z, *i.e.*, $Y \cap Z = \{0\}$ and $X = Y + Z$. Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on X that are equivalent on Y and on Z. Must they be equivalent on X?

13. Let Y be a proper closed subspace of a normed space X. Is there always a non-zero vector $x \in X$ that is 'orthogonal' to Y in the sense that $\|x + y\| \geqslant \|y\|$ for all $y \in Y$?

$^+$14. Construct two normed spaces X, Y such that $d(X, Y) = 1$ but X and Y are not isometrically isomorphic.