1. How many different partial orders (up to isomorphism) are there on a set of 4 elements? How many of these are complete?

2. Which of the following posets (ordered by inclusion) are complete?

(i) the set of all subsets of $\mathbb{N}$ that are finite or have finite complement

(ii) the set of all linearly independent subsets of a vector space V

(iii) the set of all subspaces of a vector space V

3. Let X be a complete poset, and let $f\colon X \to X$ be order-reversing (meaning that $x \leqslant y$ implies $f(x) \geqslant f(y)$). Give an example to show that f need not have a fixed point. Show, however, that there must exist either a fixed point of f or two distinct points x and y with $f(x) = y$ and $f(y) = x$.

4. Use Zorn's Lemma to show that every partial order on a set extends to a total order.

5. Give a direct proof of Zorn's Lemma (not using ordinals and not using the Axiom of Choice) for countable posets.

6. Show that the statement 'for any sets X and Y, either X injects into Y or Y injects into X' is equivalent to the Axiom of Choice (in the presence of the other rules for building sets). [Hint for one direction: Hartogs' Lemma.]

7. Formulate first-order theories in suitable languages (to be specified) whose models are precisely the following mathematical objects.

(i) fields of characteristic 2

(ii) posets having no maximal element

(iii) bipartite graphs

(iv) algebraically closed fields

(v) groups of order 60

(vi) simple groups of order 60

(vii) real vector spaces

8. Show that the sentence $(\forall\, x)(\neg(x = 0) \Rightarrow (\exists\, y)(x = sy))$ is provable in Peano Arithmetic.

9. Write down a theory (in the language of partial orders) whose models are the total orders that are dense (between any two elements, there is a third) and have no greatest or least element. Show that every countable model of this theory is isomorphic to $\mathbb{Q}$. Why does it follow that this theory is complete?

10. Show that the theory of fields of positive characteristic is not axiomatisable (in the language of rings with 1), and that the theory of fields of characteristic zero is axiomatisable but not finitely axiomatisable (*i.e.*, not axiomatisable by finitely many sentences).

11. Is every countable model of Peano Arithmetic isomorphic to $\mathbb{N}$?

12. Let L be the language consisting of a single operation symbol f of arity 1. Write down a theory T that asserts that f is a bijection with no finite orbits, and describe the countable models of T. Prove that T is a complete theory.

13. Show that every real polynomial of degree n can be written as the sum of $n+1$ periodic functions. (A function $f\colon \mathbb{R} \to \mathbb{R}$ is *periodic* if $f(x+r) = f(x)$ for some $r > 0$ and all x. Note that $n+1$ is optimal: *cf.* Part IB Tripos question 2024/2/II/8G.)

14. For $0 < a \leqslant b$, an (a,b)-*net* in a metric space (M,d) is a subset N of M such that $d(x,y) \geqslant a$ for $x \neq y$ in N and for all $x \in M$ there exists $y \in N$ such that $d(x,y) < b$. A *net* in M is an (a,b)-net for some $0 < a \leqslant b$. Show that every metric space has a net.

Let M, N be nets in an infinite-dimensional Banach space. Show that there is a bijection $f\colon M \to N$. $^{+}$Show further that there is a bijection $f\colon M \to N$ such that both f and f^{-1} are Lipschitz.