1. Write down subsets of the reals that have order-types $\omega + \omega$, $\omega^2$ and $\omega^3$.

2. Let $S$ be a set of ordinals. Show that $\alpha \cdot \sup S = \sup\{\alpha\beta : \beta \in S\}$ for any ordinal $\alpha$.

3. Show that the inductive and the synthetic definitions of ordinal multiplication coincide.

4. Let $\alpha, \beta, \gamma$ be ordinals. Prove that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$. Must we have $(\alpha+\beta)\gamma = \alpha\gamma+\beta\gamma$? Must we have $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$?

5. Is there a non-zero ordinal $\alpha$ with $\alpha\omega = \alpha$? What about $\omega\alpha = \alpha$?

6. Which of the following identities are valid for all ordinals $\alpha, \beta, \gamma$ and which might be false? Give proofs or counterexamples as appropriate.

$$\text{(i) } \alpha^{\beta+\gamma} = \alpha^\beta\alpha^\gamma \qquad \text{(ii) } \alpha^\gamma\beta^\gamma = (\alpha\beta)^\gamma \qquad \text{(iii) } \left(\alpha^\beta\right)^\gamma = \alpha^{\beta\gamma}$$

7. Let $\alpha$ and $\beta$ be ordinals with $\alpha \geqslant \beta$. Show that there is a unique ordinal $\gamma$ such that $\beta + \gamma = \alpha$. Must there exist an ordinal $\gamma$ with $\gamma + \beta = \alpha$?

8. Find two totally ordered sets such that neither is isomorphic to a subset of the other. Can you find three such sets?

9. Let $\alpha$ be a countable (non-zero) limit ordinal. Prove that there exists an increasing sequence $\alpha_1 < \alpha_2 < \alpha_3 < \dots$ with supremum equal to $\alpha$. Is this result true for $\alpha = \omega_1$?

10. Show that, for every countable ordinal $\alpha$, there is a subset of $\mathbb{Q}$ of order-type $\alpha$. Why is there no subset of $\mathbb{R}$ of order-type $\omega_1$?

11. An ordinal written as $\omega^{\alpha_1}n_1 + \cdots + \omega^{\alpha_k}n_k$, where $\alpha_1 > \cdots > \alpha_k$ are ordinals (and $k$ and $n_1, \dots, n_k$ are non-zero natural numbers), is said to be in Cantor Normal Form. Show that every non-zero ordinal has a unique Cantor Normal Form. What is the Cantor Normal Form for the ordinal $\varepsilon_0$?

12. What is the smallest fixed point of $\alpha \mapsto \omega^\alpha$? The next smallest? And the next smallest? Show that the fixed points are unbounded, and explain why this means that we may index the fixed points by the ordinals. Is there a countable ordinal $\alpha$ such that $\alpha$ is the $\alpha^{\text{th}}$ fixed point?

13. Is it possible to select for each countable (non-zero) limit ordinal $\alpha$ an ordinal $x_\alpha < \alpha$ in such a way that the $x_\alpha$ are distinct?

14. For finite subsets $A, B$ of $\mathbb{N}$, write $A \prec B$ if $A$ is a proper initial segment of $B$ (in the usual order). A family $\mathcal{A}$ of finite subsets of $\mathbb{N}$ is called *thin* if no member of $\mathcal{A}$ is a proper initial segment of another member of $\mathcal{A}$. Show that if each member of a thin family $\mathcal{A}$ is coloured 'red' or 'blue', then for some infinite subset $M$ of $\mathbb{N}$, all members of $\mathcal{A}$ contained in $M$ have the same colour. (Hint: Prove this first for thin families of countable rank where rank is defined below.)

For a family $\mathcal{A}$ of finite subsets of $\mathbb{N}$ and for $n \in \mathbb{N}$, we let

$$\mathcal{A}_n = \{A \subset \mathbb{N} : A \text{ finite}, \ n < \min A, \ \{n\} \cup A \in \mathcal{A}\}$$

where $\min \emptyset = \infty$. We then recursively define a collection $\mathcal{N}_\alpha$ of families of finite subsets of $\mathbb{N}$ by setting $\mathcal{N}_0 = \big\{\emptyset, \{\emptyset\}\big\}$ and

$$\mathcal{N}_\alpha = \big\{\mathcal{A} : \forall\, n \in \mathbb{N}\ \exists\, \beta < \alpha\ \mathcal{A}_n \in \mathcal{N}_\beta\big\}$$

for a non-zero ordinal $\alpha < \omega_1$. The *rank* of a family $\mathcal{A}$ is the least $\alpha$ such that $\mathcal{A} \in \mathcal{N}_\alpha$ if such $\alpha$ exists, and $\omega_1$ otherwise.