1. Let X be a normed space. Show that $\overline{\mathrm{span}}^{w^*} B = \big(B_\perp\big)^\perp$ for $B \subset X^*$. Deduce that a w^*-closed subspace Y of X^* is a dual space: there is a normed space Z such that $Y \cong Z^*$. Prove also the following for a bounded linear map T between normed spaces.

(i) $\ker T = (\operatorname{im} T^*)_\perp$ and $\ker T^* = (\operatorname{im} T)^\perp$.

(ii) $\overline{\operatorname{im} T} = (\ker T^*)_\perp$ and $\overline{\operatorname{im} T^*}^{w^*} = (\ker T)^\perp$.

2. Let $x_n \xrightarrow{w} 0$ in a Banach space X. Show that for all $\varepsilon > 0$ and for all $m \in \mathbb{N}$ there exists $n > m$ such that for all $x^* \in B_{X^*}$ there exists $i \in \mathbb{N}$ such that $m < i < n$ and $|x^*(x_i)| < \varepsilon$.

3. Let X be a normed space and K be a weakly compact subset of X. Show that if X^* is w^*-separable, then (K, w) is metrizable.

4. Let Z be a subspace of X^* that separates the points of X. Show that if B_X is $\sigma(X, Z)$-compact, then X is a dual space. More precisely, $X \cong Z^*$.

5. Given $F \subset X^*$ of $\dim F < \infty$, $\varphi \in B_{X^{**}}$ and $\varepsilon > 0$, show that there exists $x \in X$ with $\|x\| < 1 + \varepsilon$ such that $\hat{x}$ and φ agree on F. Use this to give an alternative proof of Goldstine's theorem.

6. Show that every weakly compact subset of ℓ_∞ is norm separable.

7. Let X be a Banach space and Y a closed subspace of X. Use the characterization of reflexivity in terms of weak compactness of the unit ball to show that if X is reflexive, then Y and X/Y are also reflexive.

8. Let $T\colon X \to Y$ be a linear bijection between (real) vector spaces and let C be a convex subset of X. Show that $T(\operatorname{Ext} C) = \operatorname{Ext} T(C)$. Now assume that $T\colon X \to Y$ is a continuous linear map between locally convex spaces and C is a compact convex subset of X. Show that $T(\operatorname{Ext} C) \supset \operatorname{Ext} T(C)$.

9. Prove that $\operatorname{Ext} B_{C(K)^*} = \{\lambda\delta_k : |\lambda| = 1,\ k \in K\}$, where K is a compact Hausdorff space.

10. Show that $B_{\ell_\infty} = \overline{\operatorname{conv}}\operatorname{Ext} B_{\ell_\infty}$ but that $B_{C[0,1]^*} \neq \overline{\operatorname{conv}}\operatorname{Ext} B_{C[0,1]^*}$.

11. Show that a weakly compact subset of a normed space is weakly sequentially compact. (See Question 18 below.)

12. Let A be a Banach algebra. Let $p \in A$ be such that $p^2 = p$ (such an element of called an *idempotent*). Show that if p is in the closure of an ideal J of A, then p does in fact belong to J.

13. Let Δ be the closed unit disc in $\mathbb{C}$. Show that one can choose a sequence (D_n) of non-overlapping open discs in the interior of Δ such that $\sum_n \operatorname{diam}(D_n) < \infty$ and $\bigcup_n D_n$ is dense in Δ. Set $K = \Delta \setminus \bigcup_n D_n$ (the *Swiss Cheese*). Show that the following formula defines a non-zero, bounded linear functional on $C(K)$.

$$\theta(f) = \int_{\partial\Delta} f(z)\,\mathrm{d}z - \sum_{n=1}^{\infty} \int_{\partial D_n} f(z)\,\mathrm{d}z$$

Deduce that $R(K) \neq A(K)$.

Some more questions

14. Let $T\colon X \to Y$ be a linear map between normed spaces. Show that if T^* is w^*-to-norm coninuous, then T is compact. Show that the converse is false. Show that in fact T is compact if and only if T^* is w^*-to-norm coninuous on w^*-compact subsets of Y^*.

15. Assume (x_n) is weakly convergent in an infinite-dimensional Banach space. Show that $\overline{\operatorname{conv}}\{x_n : n \in \mathbb{N}\}$ has empty interior. Show that this is not necessarily true for a w^*-convergent sequence in a dual space.

16. Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Assume that $T(X)$ is closed in Y. Show that if $T^{**}(\varphi) \in T(X)$ for some $\varphi \in X^{**}$, then for all $\varepsilon > 0$ there exists $x \in X$ such that $T(x) = T^{**}(\varphi)$ and $\|x\| \leqslant (1+\varepsilon)\|\varphi\|$. Show that the assumption that $T(X)$ is closed is necessary.

17. Show that every separable metric space embeds isometrically into $C[0,1]$.

18. Prove the Eberlein–Šmulian theorem: a subset of a normed space is weakly compact if and only if it is weakly sequentially compact.

19. A linear map $T\colon X \to Y$ between Banach spaces is *weakly compact* if $\overline{TB_X}$ is weakly compact. (Note that such a linear map is bounded.) Show that for $T \in \mathcal{B}(X, Y)$ the following are all equivalent to T being weakly compact.

(i) $T^{**}(X^{**}) \subset Y$;

(ii) $T^*\colon Y^* \to X^*$ is w^*-w continuous;

(iii) T^* is weakly compact.

20. Define a topology σ on X^* by declaring a set $U \subset X^*$ σ-open if and only if $U \cap S$ is an open subset of S in the relative w^*-topology of S for every bounded subset S of X^*. Show that if U is a σ-neighbourhood of 0 then there is a null-sequence (x_n) in X such that

$$\big\{x^* \in X^* : |x^*(x_n)| \leqslant 1 \text{ for all } n \in \mathbb{N}\big\} \subset U \ .$$

Deduce the following consequences.

(i) The dual spaces of (X^*, σ) and (X^*, w^*) are the same.

(ii) $\phi \in X^{**}$ is w^*-continuous if and only if $\phi\restriction_{B_{X^*}}$ is w^*-continuous.

(iii) A subspace Y of X^* is w^*-closed if and only if B_Y is w^*-compact.