

# Resumé on Hilbert spaces and Spectral Theory

## 1 Hilbert spaces

I assume you know what a Hilbert space is and that you are familiar with basic Hilbert space geometry (Parallelogram Law, orthogonality, Pythagoras's Theorem, orthogonal complements, etc). We recall the following result.

**1.1** If  $Y$  is a closed subspace of a Hilbert space  $H$ , then  $H$  is the orthogonal direct sum of  $Y$  and  $Y^\perp$  (written  $H = Y \oplus Y^\perp$ ).

**1.2** Corresponding to the orthogonal decomposition  $Y \oplus Y^\perp$  of  $H$  is the map

$$P: H \rightarrow H \quad \text{given by} \quad P(y + z) = y \quad (y \in Y, z \in Y^\perp).$$

This is a bounded linear map with  $\text{im } P = Y$  and  $\text{ker } P = Y^\perp$ . It is called the *orthogonal projection* of  $H$  onto  $Y$ .

**1.3 Riesz Representation Theorem** (Yes, another one.) For each  $f \in H^*$  there is a unique  $y \in H$  such that  $f(x) = \langle x, y \rangle$  for all  $x \in H$ . The map  $f \mapsto y$  is an isometric, conjugate-linear isomorphism of  $H^*$  onto  $H$ .

**1.4** A *sesquilinear form* on a (complex) Hilbert space  $H$  is a map  $\theta: H \times H \rightarrow \mathbb{C}$  satisfying for all  $x, y, z \in H$  and  $\lambda, \mu \in \mathbb{C}$  that

- (i)  $\theta(\lambda x + \mu y, z) = \lambda \theta(x, z) + \mu \theta(y, z)$  (linearity in first variable), and
- (ii)  $\theta(x, \lambda y + \mu z) = \bar{\lambda} \theta(x, y) + \bar{\mu} \theta(x, z)$  (conjugate-linearity in second variable).

The sesquilinear form  $\theta$  is called a *hermitian form* if in addition it satisfies

- (iii)  $\theta(y, x) = \overline{\theta(x, y)}$  for all  $x, y \in H$ .

Note that (i) and (iii) imply (ii). A sesquilinear form  $\theta$  is *bounded* if there is a constant  $C \geq 0$  such that

$$|\theta(x, y)| \leq C \|x\| \|y\| \quad \text{for all } x, y \in H.$$

This is equivalent to  $\theta$  being continuous. The smallest  $C$  that works is denoted by  $\|\theta\|$ . Note that

$$\|\theta\| = \sup \{ |\theta(x, y)| : x, y \in H, \|x\| \leq 1, \|y\| \leq 1 \}.$$

*E.g.*, the inner product  $\langle \cdot, \cdot \rangle$  is a bounded hermitian form on  $H$  with norm 1.

**1.5 Theorem** Let  $\theta$  be a bounded sesquilinear form on  $H$ . Then there is a unique map  $T: H \rightarrow H$  such that

$$(1) \quad \theta(x, y) = \langle Tx, y \rangle \quad \text{for all } x, y \in H.$$

Moreover,  $T \in \mathcal{B}(H)$  and  $\|T\| = \|\theta\|$ .

*Proof.* Fix  $x \in H$ . The map  $y \mapsto \overline{\theta(x, y)}$  is a bounded linear map of norm at most  $\|\theta\| \|x\|$ . By the Riesz Representation Theorem, there exists some  $Tx \in H$  such that  $\overline{\theta(x, y)} = \langle y, Tx \rangle$  for all  $y \in H$ . This defines a map  $T: H \rightarrow H$  satisfying (1). Given  $x, y, z \in H$  and  $\lambda, \mu \in \mathbb{C}$ , we have

$$\begin{aligned} \langle T(\lambda x + \mu y), z \rangle &= \theta(\lambda x + \mu y, z) = \lambda \theta(x, z) + \mu \theta(y, z) \\ &= \lambda \langle Tx, z \rangle + \mu \langle Ty, z \rangle \\ &= \langle \lambda Tx + \mu Ty, z \rangle \end{aligned}$$

Since  $z$  was arbitrary, it follows that  $T(\lambda x + \mu y) = \lambda Tx + \mu Ty$ , and  $T$  is linear. Next,

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \theta(x, Tx) \leq \|\theta\| \|x\| \|Tx\| .$$

Hence  $T$  is bounded with  $\|T\| \leq \|\theta\|$ . Conversely,

$$|\theta(x, y)| = |\langle Tx, y \rangle| \leq \|Tx\| \|y\| \leq \|T\| \|x\| \|y\|$$

by Cauchy-Schwarz. Thus  $\|\theta\| = \|T\|$ .

Finally, to show uniqueness, assume that  $\langle Tx, y \rangle = \langle Sx, y \rangle$  for all  $x, y$ . Putting  $y = Tx - Sx$  yields  $\|Tx - Sx\| = 0$ , and hence  $T = S$ .  $\square$

**1.6 Adjoins** For  $T \in \mathcal{B}(H)$  there is a unique map  $T^*: H \rightarrow H$  such that  $\langle Tx, y \rangle = \langle x, T^*y \rangle$  for all  $x, y \in H$ . Moreover,  $T^* \in \mathcal{B}(H)$  and  $\|T^*\| = \|T\|$ .

*Proof.* Apply Theorem 1.5 to  $\theta(x, y) = \langle x, Ty \rangle$ .  $\square$

**1.7 Remark**  $T^*: H \rightarrow H$  is called the *adjoint* of  $T$ . By the Riesz Representation Theorem,  $T^*$  can be viewed as a map  $H^* \rightarrow H^*$ , and then it is nothing else but the dual operator of  $T$ .

**1.8 Properties of adjoints** Let  $S, T \in \mathcal{B}(H)$  and  $\lambda, \mu \in \mathbb{C}$ .

- (i)  $(\lambda S + \mu T)^* = \bar{\lambda} S^* + \bar{\mu} T^*$
- (ii)  $(ST)^* = T^* S^*$
- (iii)  $T^{**} = T$
- (iv)  $\|T^* T\| = \|T\|^2$

**1.9** An operator  $T \in \mathcal{B}(H)$  is

- (i) *hermitian* if  $T^* = T$
- (ii) *unitary* if  $TT^* = T^*T = I$
- (iii) *normal* if  $TT^* = T^*T$ .

Examples of hermitian operators include orthogonal projections. An operator is unitary if and only if it is isometric and surjective. Examples of normal operators include all hermitian and unitary operators.

**1.10 Note** If  $\theta$  in Theorem 1.5 is hermitian, then the corresponding operator  $T$  is also hermitian.

## 2 Spectral Theory

Let  $X$  be a (non-zero) complex Banach space and  $T \in \mathcal{B}(X)$ . The *spectrum* of  $T$  is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ not invertible}\} .$$

This is a special case of the spectrum of an element of a unital Banach algebra as defined in the course. In particular, the spectrum of  $T$  is a non-empty, compact subset of  $\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$ . Moreover the *spectral radius*  $r(T)$  of  $T$  satisfies

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{n \rightarrow \infty} \|T^n\|^{1/n} .$$

**2.1** We say  $\lambda$  is an *approximate eigenvalue* of  $T$  if there exists a sequence  $(x_n)$  in  $X$  with  $\|x_n\| = 1$  for all  $n \in \mathbb{N}$  such that

$$(\lambda I - T)x_n \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

The sequence  $(x_n)$  is an *approximate eigenvector* for  $\lambda$ . The *approximate point spectrum* of  $T$  is the set of all approximate eigenvalues of  $T$ , and is denoted by  $\sigma_{\text{ap}}(T)$ . We also define the *point spectrum* of  $T$  to be the set of all eigenvalues of  $T$  and denote it by  $\sigma_{\text{p}}(T)$ . One clearly has

$$\sigma_{\text{p}}(T) \subset \sigma_{\text{ap}}(T) \subset \sigma(T) .$$

In general, these inclusions can be strict and the point spectrum can be empty (unlike the spectrum). However, we have the following result. Here  $\partial A$  denotes the boundary of a set  $A$  in a topological space  $X$ :  $\partial A = \bar{A} \setminus A^\circ$ .

**2.2 Theorem** We have  $\partial\sigma(T) \subset \sigma_{\text{ap}}(T)$ . In particular,  $\sigma_{\text{ap}}(T) \neq \emptyset$ .

*Proof.* Let  $\lambda \in \partial\sigma(T)$ . Then there is a sequence  $\lambda_n \notin \sigma(T)$  converging to  $\lambda$ . It follows from a result proved in lectures that

$$\|(\lambda_n I - T)^{-1}\| \rightarrow \infty \quad \text{as } n \rightarrow \infty .$$

Thus, there is a sequence  $(x_n)$  of unit vectors such that

$$\|(\lambda_n I - T)^{-1}x_n\| \rightarrow \infty \quad \text{as } n \rightarrow \infty .$$

Set

$$y_n = \frac{(\lambda_n I - T)^{-1}x_n}{\|(\lambda_n I - T)^{-1}x_n\|} .$$

It is easy to check that  $(y_n)$  is an approximate eigenvector for  $\lambda$ . □

**2.3 Theorem** Let  $T \in \mathcal{B}(X)$  be a compact operator. Let  $\lambda \in \sigma_{\text{ap}}(T)$  and  $\lambda \neq 0$ . Then  $\lambda$  is an eigenvalue of  $T$ .

**2.4** From now on  $H$  is a (non-zero) complex Hilbert space. For  $T \in \mathcal{B}(H)$  we have

$$\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\} .$$

If  $T$  is hermitian then  $\sigma(T) \subset \mathbb{R}$ . It follows that  $\sigma(T) = \sigma_{\text{ap}}(T)$ . This latter fact holds also when  $T$  is a normal operator.

**2.5 Theorem** Let  $T \in \mathcal{B}(H)$  be a compact hermitian operator. Then  $\sigma(T)$  is countable and if  $\lambda \in \sigma(T)$ ,  $\lambda \neq 0$ , then  $\lambda$  is an eigenvalue whose eigenspace is finite-dimensional:  $\dim \ker(\lambda I - T) < \infty$ .

**2.6 Spectral Theorem** Let  $T \in \mathcal{B}(H)$  be a compact hermitian operator. Then there is an orthonormal sequence  $x_1, x_2, \dots$  (finite or infinite) of eigenvectors of  $T$  with non-zero eigenvalues  $\lambda_1, \lambda_2, \dots$ , respectively, such that

$$T\left(\sum a_n x_n + z\right) = \sum \lambda_n a_n x_n$$

for all scalars  $a_n$ , and all  $z \in \{x_n : n \in \mathbb{N}\}^\perp$ .

**2.7 Remark** The above holds for compact normal operators as well.

**2.8 Definition**  $T \in \mathcal{B}(H)$  is *positive* if  $T$  is hermitian and  $\sigma(T) \subset \mathbb{R}^+$ .

**2.9 Theorem**  $T \in \mathcal{B}(H)$  is positive if and only if  $\langle Tx, x \rangle \geq 0$  for all  $x \in H$ .