

Measure Theory

We list here definitions and results from basic measure theory. These can be found in any good book on measure theory, *e.g.*, the ones by S. J. Taylor or D. L. Cohn.

1 Measures

1.1 A *measure space* is a triple $(\Omega, \mathcal{F}, \mu)$, where

- (i) Ω is a set;
- (ii) \mathcal{F} is a σ -field on Ω , *i.e.*, $\mathcal{F} \subset \mathcal{P}(\Omega)$ such that
 - (a) $\emptyset \in \mathcal{F}$,
 - (b) if $A \in \mathcal{F}$ then $\Omega \setminus A \in \mathcal{F}$,
 - (c) if $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$;
- (iii) $\mu: \mathcal{F} \rightarrow [0, \infty]$ is a *measure on \mathcal{F}* :
 - (a) $\mu(\emptyset) = 0$,
 - (b) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ for pairwise disjoint sets A_n , $n \in \mathbb{N}$, in \mathcal{F} .

Here we use the usual conventions regarding ∞ . *E.g.*, $x + \infty = \infty + x = \infty$ for all $x \in \mathbb{R}$.

1.2 A measure μ is *finite* if $\mu(\Omega) < \infty$. In this case $\mu(A) < \infty$ for all $A \in \mathcal{F}$.

1.3 $N \in \mathcal{F}$ is called a *null set* (or μ -*null set*) if $\mu(N) = 0$.

1.4 If \mathcal{A} is an arbitrary family of subsets of a set Ω , then there is a (unique) smallest σ -field on Ω containing \mathcal{A} , which is the intersection of all σ -fields on Ω that contain \mathcal{A} . It is called the σ -field *generated by \mathcal{A}* .

1.5 Let X be a topological space. We denote by \mathcal{G} the family of open subsets of X . The *Borel σ -field on X* is the σ -field \mathcal{B} generated by \mathcal{G} . Elements of \mathcal{B} are called *Borel sets*. A *Borel measure on X* is a measure on \mathcal{B} .

2 Outer measures

2.1 Given a set Ω , an *outer measure on Ω* is a function $\mu^*: \mathcal{P}(\Omega) \rightarrow [0, \infty]$ such that

- (i) $\mu^*(\emptyset) = 0$
- (ii) $\mu^*(A) \leq \mu^*(B)$ whenever $A \subset B$
- (iii) $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ for arbitrary subsets A_n of Ω .

2.2 $A \subset \Omega$ is called μ^* -measurable if

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A)$$

holds for all $B \subset \Omega$.

2.3 Theorem The family \mathcal{M} of μ^* -measurable subsets of Ω is a σ -field on Ω , and the restriction μ of μ^* to \mathcal{M} is a measure on \mathcal{M} .

3 Measurable functions

3.1 Let Ω be a set and \mathcal{F} be a σ -field on Ω . A function $f: \Omega \rightarrow \mathbb{R}$ (or \mathbb{C}) is *measurable* if $f^{-1}(B) \in \mathcal{F}$ for every Borel set $B \subset \mathbb{R}$ (respectively, \mathbb{C}).

3.2 Examples

- (i) If Ω is a topological space, \mathcal{F} is the Borel σ -field on Ω , and f is a continuous scalar-valued function on Ω , then f is measurable.
- (ii) In general, any *simple function*, *i.e.*, a function of the form $\sum_{k=1}^n a_k \mathbf{1}_{A_k}$ where $A_k \in \mathcal{F}$ and a_k is a scalar for all $1 \leq k \leq n$, is measurable.

3.3 The set of all measurable functions on Ω is an algebra under pointwise operations. If $f: \Omega \rightarrow \mathbb{C}$ is measurable, then so are $|f|$, the real part $\mathcal{R}(f)$ of f , and the imaginary part $\mathcal{I}(f)$ of f . If $f, g: \Omega \rightarrow \mathbb{R}$ are measurable, then so are their maximum $f \vee g$, and their minimum $f \wedge g$. Finally, if (f_n) is a sequence of measurable functions that converges pointwise to a function f , then f is measurable.

4 Integration

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. We define $\int_{\Omega} f \, d\mu$ for *certain* scalar-valued, measurable functions on Ω .

4.1 If $f \geq 0$ is a simple function, *i.e.*, $f = \sum_{k=1}^n a_k \mathbf{1}_{A_k}$ where $A_k \in \mathcal{F}$ and $a_k \geq 0$ for all $1 \leq k \leq n$, then we define

$$\int_{\Omega} f \, d\mu = \sum_{k=1}^n a_k \mu(A_k)$$

which is a number in $[0, \infty]$. We use the convention $0 \cdot \infty = \infty \cdot 0 = 0$.

4.2 If $f \geq 0$ is measurable, then we let

$$\int_{\Omega} f \, d\mu = \sup \left\{ \int_{\Omega} g \, d\mu : 0 \leq g \leq f, g \text{ a simple function} \right\}$$

which is again a number in $[0, \infty]$.

4.3 $f: \Omega \rightarrow \mathbb{R}$ is called *integrable* if it is measurable and $\int_{\Omega} |f| \, d\mu$ is finite. We then set

$$\int_{\Omega} f \, d\mu = \int_{\Omega} f^+ \, d\mu - \int_{\Omega} f^- \, d\mu$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

4.4 $f: \Omega \rightarrow \mathbb{C}$ is called *integrable* if it is measurable and $\int_{\Omega} |f| \, d\mu$ is finite. We then set

$$\int_{\Omega} f \, d\mu = \int_{\Omega} \mathcal{R}(f) \, d\mu + i \cdot \int_{\Omega} \mathcal{I}(f) \, d\mu$$

where $\mathcal{R}(f)$ and $\mathcal{I}(f)$ are the real and imaginary parts of f , respectively.

4.5 Properties

(i) Linearity:

(a) If $f \geq 0$, $g \geq 0$ are measurable, and $\alpha \geq 0$, $\beta \geq 0$ are real numbers, then

$$\int_{\Omega} (\alpha f + \beta g) \, d\mu = \alpha \cdot \int_{\Omega} f \, d\mu + \beta \cdot \int_{\Omega} g \, d\mu .$$

(b) If f, g are integrable functions and α, β are scalars, then $\alpha f + \beta g$ is integrable and

$$\int_{\Omega} (\alpha f + \beta g) \, d\mu = \alpha \cdot \int_{\Omega} f \, d\mu + \beta \cdot \int_{\Omega} g \, d\mu .$$

(ii) Monotone convergence: if $0 \leq f_n \nearrow f$ pointwise a.e. (almost everywhere), then $\int_{\Omega} f_n \, d\mu \nearrow \int_{\Omega} f \, d\mu$.

(iii) Fatou's lemma: if (f_n) is a sequence of measurable functions such that $f_n \geq g$ for all $n \in \mathbb{N}$ for some integrable function g , then

$$\int_{\Omega} \liminf f_n \, d\mu \leq \liminf \int_{\Omega} f_n \, d\mu .$$

(iv) Dominated convergence: if f_n ($n \in \mathbb{N}$), f and g are measurable functions such that $|f_n| \leq g$ for all $n \in \mathbb{N}$, $f_n \rightarrow f$ pointwise a.e., and g is integrable, then f is integrable and $\int_{\Omega} f_n \, d\mu \rightarrow \int_{\Omega} f \, d\mu$.

4.6 A property of points of Ω is said to hold *almost everywhere* (or *a.e.* for short) if it holds for all $\omega \in \Omega \setminus N$ for some null set $N \in \mathcal{F}$. We sometimes use the term μ -almost everywhere (or μ -a.e. for short) to emphasize the measure μ .

5 L_p spaces

Throughout this section, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

5.1 Let $1 \leq p < \infty$. We define $L_p(\Omega, \mathcal{F}, \mu)$ or simply $L_p(\mu)$, to be the real (or complex) vector space of all measurable functions $f: \Omega \rightarrow \mathbb{R}$ (respectively, \mathbb{C}) such that $\int_{\Omega} |f|^p \, d\mu < \infty$.

5.2 Let $1 \leq p < \infty$. For $f \in L_p(\mu)$ we define its L_p -norm by

$$\|f\|_p = \left(\int_{\Omega} |f|^p \, d\mu \right)^{\frac{1}{p}} .$$

5.3 A measurable function $f: \Omega \rightarrow \mathbb{R}$ (or \mathbb{C}) is *essentially bounded* if there is a μ -null set $N \in \mathcal{F}$ such that f is bounded on $\Omega \setminus N$.

5.4 We define $L_\infty(\Omega, \mathcal{F}, \mu)$, or simply $L_\infty(\mu)$, to be the real (or complex) vector space of all measurable, essentially bounded functions $f: \Omega \rightarrow \mathbb{R}$ (respectively, \mathbb{C}).

5.5 For $f \in L_\infty(\mu)$ we define its *essential sup norm* or L_∞ -norm by

$$\|f\|_\infty = \text{ess sup}|f| = \inf \left\{ \sup_{\Omega \setminus N} |f| : N \in \mathcal{F}, \mu(N) = 0 \right\} .$$

Note that the essential supremum is attained: there is a μ -null set $N \in \mathcal{F}$ such that $\text{ess sup}|f| = \sup_{\Omega \setminus N} |f|$.

5.6 Theorem (Hölder) Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $f \in L_p(\mu)$ and $g \in L_q(\mu)$ we have $fg \in L_1(\mu)$ and

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q .$$

5.7 Theorem (Minkowski) Let $1 \leq p \leq \infty$ and let $f, g \in L_p(\mu)$. Then $f + g \in L_p(\mu)$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p .$$

5.8 It follows from the above that for $1 \leq p \leq \infty$ the space $L_p(\mu)$ is a normed space in the L_p -norm provided we identify functions f and g if $f = g$ a.e. (almost everywhere), *i.e.*, when $\{\omega \in \Omega : f(\omega) \neq g(\omega)\}$ is a μ -null set (has μ -measure zero).

5.8.1 Remark Strictly speaking $\|\cdot\|_p$ is a seminorm on $L_p(\mu)$ for $1 \leq p \leq \infty$. In general, if $\|\cdot\|$ is a seminorm on a real or complex vector space X , then $N = \{z \in X : \|z\| = 0\}$ is a subspace of X , and $\|x + N\| = \|x\|$ defines a norm on the the quotient space X/N . However, we will not do this for $L_p(\mu)$. We prefer to think of elements of $L_p(\mu)$ as functions rather than equivalence classes of functions. One must remember that equality in $L_p(\mu)$ means a.e. equality.

5.9 Theorem For $1 \leq p \leq \infty$, the space $L_p(\mu)$ is complete in the L_p -norm.